Leibniz algebras and dialgebras

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Abstract

This note is a brief survey on the main known results on Leibniz algebras and dialgebras, introduced by J.-L. Loday as a generalization of Lie algebras and associative algebras respectively, and on their natural homology theories. Two examples of a Leibniz algebra are given: the non-commutative Steinberg algebra introduced by J.-L. Loday and T. Pirashvili, and the extension of the Poisson bracket of functions to the algebras of multivector fields and differential forms on a Poisson manifolds, found by Y. Kosmann-Schwarzbach.

Leibniz homology produces new invariants for Lie algebras. We present some examples of these new invariants for some finite dimensional Lie algebras, for the lie algebra of vector fields on the real space, and the relationship with the rational homotopy theory of topological spaces. Finally, we present the author's contribution to the computation of Leibniz homology, based on a homology theory with coefficients for dialgebras.

Introduction

Leibniz algebras are a generalization of Lie algebras introduced by J.-L. Loday in 1989. His main motivation was the search of an 'obstruction' to the periodicity of algebraic K-theory, which does not satisfy the Bott periodicity theorems valid for its topological version. We recall that cyclic homology of associative algebras can be regarded as an 'additive' or 'infinitesimal' version of rational algebraic K-theory, by virtue of the following results,

- $\bigwedge(K_*(A)_{\mathbf{Q}}) \cong H_*^{gr}(GL(A), \mathbf{Q}) \cong H_*^{Hoch}(\mathbf{Q}[GL(A)], \mathbf{Q})$, [Quillen]
- $\bigwedge(HC_{*-1}(A)) \cong H_*^{Lie}(\mathfrak{gl}(A), \mathbf{Q})$, [Loday-Quillen-Tsygan]

which hold for any unital associative algebra A. The long exact sequence

- $\ldots \longrightarrow HH_n(A) \longrightarrow HC_n(A) \longrightarrow HC_{n-2}(A) \longrightarrow \ldots$, [Connes]

leads us to interpret Hochschild homology as an obstruction to the periodicity of cyclic homology and to ask a natural question (see [L1] for more details): is there a theory KX_* (where X means 'unknown' for the moment) analogue to HH_*, which would be part of a spectral sequence involving the theory K_* and whose additive version would be Connes' long exact sequence?

The discovery of Leibniz algebras, and of their natural homology theory HL_*, produced a new relation

- $T(HH_{*-1}(A)) \cong HL_*(\mathfrak{gl}(A), \mathbf{Q})$, [Cuvier-Loday], where T denotes the tensor functor. Then the idea is to define $KX_*(A)$ in such a way that $T(KX_*(A)_{\mathbf{Q}}) \cong HX_*(\mathbf{Q}[GL(A)], \mathbf{Q})$, and then to compare it with $K_*(A)$. Here HX_* should be a new homology theory, related to Leibniz homology by a formula $HL_*(\mathfrak{g}, U\mathfrak{g}^{ad}) \cong HHX_*(UX\mathfrak{g})$, analogue to the classical relation

- $H_*^{Lie}(\mathfrak{g}, U\mathfrak{g}^{ad}) \cong HH_*(U\mathfrak{g})$,

which recovers the homology of Lie algebras from that of associative algebras. This program lead Loday to define a suitable 'associative' version of Leibniz algebras, called dialgebras, and to construct their natural homology theory. A final answer to the original problem is not achieved yet.

Beside this purely algebraic motivation and developement of Leibniz' setting, some relationships with classical geometry have recently been discovered, which could lead us to investigate Leibniz homology in view of concrete applications in non-commutative geometry and its physical interpretations.

Contents

1 Leibniz algebras

The Chevalley-Eilenberg chain complex of a Lie algebra $\mathfrak{g}$ is the sequence of chain modules given by the exterior powers $\bigwedge^* \mathfrak{g}$ and boundary operators $d : \bigwedge^n \mathfrak{g} \longrightarrow \bigwedge^{n-1} \mathfrak{g}$ classically defined as

$$d(x_1 \wedge \ldots \wedge x_n) := \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n.$$

The property $d \circ d = 0$, which makes this sequence a chain complex, is proved by using the antisymmetry $x \wedge y = -y \wedge x$ of the exterior product, the Jacobi identity $[[x,y],z] + [[y,z],x] + [[z,x],y] = 0$ and the antisymmetry $[x,y] = -[y,x]$ of the Lie bracket given on $\mathfrak{g}$.

In 1989, J.-L. Loday remarked that if we rewrite the Chevalley-Eilenberg boundary operator as

$$d(x_1 \wedge \ldots \wedge x_n) = \sum_{i<j} (-1)^{j+1} x_1 \wedge \ldots \wedge x_{i-1} \wedge [x_i, x_j] \wedge x_{i+1} \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n,$$

then the property $d \circ d = 0$ is proved *without* making use of the antisymmetry properties of both the exterior product and of the Lie bracket on $\mathfrak{g}$: it suffices that the bracket satisfies the so called *Leibniz identity*

$$[x, [y, z]] = [[x, y], z] - [[x, z], y],$$

which is equivalent to the Jacobi identity any time the bracket is also antisymmetric.

From these considerations follow some facts:

- the Chevalley-Eilenberg boundary operator (in the Loday form) can be lifted to the *tensor powers* $\mathfrak{g}^{\otimes *}$ of a Lie algebra $\mathfrak{g}$, giving rise to a new chain complex $d : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}^{\otimes n-1}$ (called *Leibniz complex*);

- Leibniz homology indeed provides *a new family of invariants for Lie algebras*. It suffices to see it for an abelian Lie algebra $\mathfrak{g}$, which is a Lie algebra with zero bracket $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. All boundary operators are canonically zero, and yield the homology groups $H_n(\mathfrak{g}) = \bigwedge^n(\mathfrak{g})$ and $HL_n(\mathfrak{g}) = \mathfrak{g}^{\otimes n}$.

- the Leibniz complex is well defined not only for Lie algebras, but more generally for the larger class of algebras whose bracket satisfies the Leibniz identity but is not necessarily antisymmetric. These algebras are called *Leibniz algebras*.

In conclusion, Loday introduced a new category of algebras:

1.1 - Leibniz algebras. A *Leibniz algebra* is a vector space $\mathfrak{g}$ equipped with a binary operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies the *Leibniz identity*

$$(L) \quad [x, [y, z]] = [[x, y], z] - [[x, z], y], \qquad x, y, z \in \mathfrak{g}.$$

Lie algebras are a trivial example of Leibniz algebras, those with antisymmetric bracket.

Some remarks on the name and the meaning of Leibniz identity may be helpful.

- It is custom to call "Leibniz rule" the property which characterizes the derivation of a product, namely $D([a, b]) = [D(a), b] + [a, D(b)]$. The Leibniz identity given above has in fact the same origin, since it says that *the adjoint operator* $\mathrm{ad}_x = [-, x]$ *is a derivation of the Leibniz bracket*.

- The Leibniz identity appears in the context of Lie algebras as the condition for which *any Lie algebra is a right module over itself*, when $x \in \mathfrak{g}$ is understood as an element of the module, and $y, z \in \mathfrak{g}$ as elements of the Lie algebra acting on the module from the right.

Here are two non-trivial examples of Leibniz algebras.

1.2 - The non-commutative Steinberg algebra. Let A be an associative unital algebra, and $n \geq 3$ a natural number. The *non-commutative Steinberg algebra* $\mathfrak{stl}_n(A)$ is the Leibniz algebra generated by the linear symbols $v_{ij}(a)$, for $a \in A$ and $1 \leq i, j \leq n$ with $i \neq j$, with bracket

$$[v_{ij}(a), v_{kl}(b)] = \begin{cases} 0, & \text{if } i \neq j \text{ and } j \neq l \\ v_{ik}(ab), & \text{if } i \neq j \text{ and } j = l \\ -v_{lj}(ba), & \text{if } i = j \text{ and } j \neq l \end{cases} .$$

Loday and Pirashvili, in [LP], proved the following facts.

- For $n \geq 3$, the algebra $\mathfrak{stl}_n(A)$ is the universal extension of $\mathfrak{sl}_n(A)$ in the category of Leibniz algebras, where $\mathfrak{sl}_n(A)$ is the Lie algebra of elementary matrices $E_{ij}(a)$ and the extension is given by the map
$$\varphi : \mathfrak{stl}_n(A) \longrightarrow \mathfrak{sl}_n(A), \quad v_{ij}(a) \mapsto E_{ij}(a).$$

- For $n \geq 5$, the following diagram of central extensions is commutative:

$$\begin{tikzcd}
0 \arrow[r] & HH_1(A) \arrow[r] \arrow[d] & \mathfrak{stl}_n(A) \arrow[r] \arrow[d] & \mathfrak{sl}_n(A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & HC_1(A) \arrow[r] & \mathfrak{st}_n(A) \arrow[r] & \mathfrak{sl}_n(A) \arrow[r] & 0
\end{tikzcd} \quad .$$

1.3 - The derived Leibniz algebra of a differential Lie algebra. Let $\mathfrak{g}$ be a (graded) differential Lie algebra, with bracket $[\ ,\ ]$ and differential d. Then the *derived bracket*

$$[x,y]_d := [x,dy], \qquad x,y \in \mathfrak{g},$$

is a (graded) Leibniz bracket which is not antisymmetric, in general. The new algebra $(\mathfrak{g},[\ ,\ ]_d)$ is called *derived Leibniz algebra* and denoted by $\mathfrak{g}_d$.

1.4 - Leibniz brackets extending a Poisson bracket. In particular, given a Poisson manifold M, with Poisson bivector $P \in \Gamma^\infty(\bigwedge^2 TM)$, there exist two graded differential Lie algebras naturally associated to (M,P).

The first is the set of *multivector fields* $V^n(M) := \Gamma^\infty(\bigwedge^n TM)$ with the *Schouten-Nijenhuis bracket* $[\ ,\ ] : V^n(M) \otimes V^m(M) \longrightarrow V^{n+m-1}(M)$, cf. [N], [S], defined by $[f,g] = f\cdot g - g \cdot f = 0$ for $n,m=0$, $[f,X] := X(f)$ for $n=0$ and $m=1$ (or the opposite), $[X,Y] := XY - YX$ for $n,m=1$, and then extended for $n,m \geq 1$ as

$$[X_1 \wedge \ldots \wedge X_n, Y_1 \wedge \ldots \wedge Y_m] := \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge \widehat{Y}_j \wedge \ldots \wedge Y_m,$$

and the *Lichnerowicz-Poisson differential* $d_P : V^n(M) \longrightarrow V^{n+1}(M)$, cf. [Li], defined by

$$d_P(V) := [P,V], \qquad V \in V^n(M).$$

The second is the set of *differential forms* $\Omega^n(M) := \Gamma^\infty(\bigwedge^n T^*M)$ on the manifold M, with the *de Rham differential* $d : \Omega^n(M) \longrightarrow \Omega^{n+1}(M)$ and the *Koszul bracket* $[\ ,\ ]_P : \Omega^n(M) \otimes \Omega^m(M) \longrightarrow \Omega^{n+m-1}(M)$, cf. [K], defined by $[f,g]_P = f \cdot g - g \cdot f = 0$ for $n,m=0$, $[f,\omega]_P := i_P(df \wedge \omega)$ for $n=0$ and $m=1$ (or the opposite), where i_P is the contraction by the bivector P, $[\omega,\eta]_P := \delta^*(\omega\wedge\eta) - \delta^*(\omega)\wedge\eta - (-1)^{|\omega|}\omega \wedge \delta^*(\eta)$ for $n,m=1$, where $\delta^* := [d, i_P] : \Omega^n(M) \longrightarrow \Omega^{n-1}(M)$ is the boundary operator which computes the Poisson homology of M (in particular, $[df,dg]_P = d\, i_P(df \wedge dg)$), and then extended for $n,m \geq 1$ as the Schouten-Nijenhuis bracket,

$$[\omega_1 \wedge \ldots \wedge \omega_n, \eta_1 \wedge \ldots \wedge \eta_m] := \sum_{i,j} (-1)^{i+j} [\omega_i, \eta_j] \wedge \omega_1 \wedge \ldots \wedge \widehat{\omega}_i \wedge \ldots \wedge \widehat{\eta}_j \wedge \ldots \wedge \eta_m.$$

The derived brackets

- $[U,V]_{d_P} := [V, d_P U]$ on multivector fields $U,V \in V^*(M)$,
- $[\omega,\eta]_{P,d} := [\omega, d\eta]_P$ on differential forms $\omega,\eta \in \Omega^*(M)$,

define on $V^*(M)$ and $\Omega^*(M)$ the structure of a graded Leibniz algebra which is not graded Lie.

The remarkable fact is that the Poisson bracket $\{\ ,\ \} : C^\infty(M) \otimes C^\infty(M) \longrightarrow C^\infty(M)$, related to P by the formula $\{f,g\} = i_P(df,dg)$, can be recovered from

- the Schouten-Nijenhuis bracket and the Lichnerowicz-Poisson differential on multivector fields, because $\{f,g\} = -[f, d_P g]$ for any $f,g \in C^\infty(M)$;
- the Koszul bracket and the de Rham differential on differential forms, because $\{f,g\} = [f, dg]_P$ for any $f,g \in C^\infty(M)$.

2 Leibniz homology and cohomology

2.1 - Leibniz (co)homology. Let $\mathfrak{g}$ be a Leibniz algebra. Consider the sequence of chain modules and boundary operators $d : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}^{\otimes n-1}$, with

$$d(x_1 \otimes \ldots \otimes x_n) := \sum_{i<j} (-1)^{j+1} x_1 \otimes \ldots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_n.$$

The property $d \circ d = 0$ holds because of the Leibniz identity. Hence this sequence defines a chain complex whose homology is called *Leibniz homology* of $\mathfrak{g}$, and denoted by $HL_*(\mathfrak{g})$.

Of course this homology theory can be dualized into a cohomology theory, given by the sequence of cochain modules and coboundary operators $\delta : \mathrm{Hom}(\mathfrak{g}^{\otimes n}, k) \longrightarrow \mathrm{Hom}(\mathfrak{g}^{\otimes n+1}, k)$, with $\delta(f) := f \circ d$. This sequence defines a cochain complex whose homology is called *Leibniz cohomology* of $\mathfrak{g}$, and denoted by $HL^*(\mathfrak{g})$.

2.2 - The coefficients: representations and corepresentations. Unlike for Lie algebras, for Leibniz algebras it is not a priori clear which relations should be satisfied by the non-trivial coefficients of a (co)homology theory. We outline the procedure to define the suitable modules in order to obtain the main expected theorems on homology and cohomology.

- First, define a *representation* of a Leibniz algebra $\mathfrak{g}$ to be a module M on which $\mathfrak{g}$ acts with a left and a right action, both denoted by $[\,,\,]$, which satisfy relations (L) for each choice of x, y, z in M. Then it turns out that we can consider the abelian extensions of $\mathfrak{g}$ by M,
$$0 \longrightarrow M \longrightarrow \mathfrak{g}' \longrightarrow \mathfrak{g} \longrightarrow 0.$$
By choosing the suitable coboundary operator on $\mathrm{Hom}(\mathfrak{g}^{\otimes n}, M)$, one is then able to prove the expected result that $HL^2(\mathfrak{g}, M)$ classifies the abelian extensions of $\mathfrak{g}$ by M up to equivalence.

- Once the notion of $\mathfrak{g}$-representation is given, define the *universal enveloping algebra* of $\mathfrak{g}$ as the (smallest) unital associative algebra $UL(\mathfrak{g})$ such that
$$M \text{ is a } \mathfrak{g}\text{-representation} \qquad \Longleftrightarrow \qquad M \text{ is a } UL(\mathfrak{g})\text{-left unital module.}$$

- Finally, define a *corepresentation* N of $\mathfrak{g}$ as a $UL(\mathfrak{g})$-right unital module. It turns out that the module N does not satisfies the same relations (L), but some other relations which can be found in [LP].

2.3 - Leibniz (co)homology with coefficients. Let $\mathfrak{g}$ be a Leibniz algebra. Given a corepresentation N of $\mathfrak{g}$, it is possiblle to extend the Leibniz boundary operator to the sequence $d : N \otimes \mathfrak{g}^{\otimes n} \longrightarrow N \otimes \mathfrak{g}^{\otimes n-1}$, with

$$\begin{aligned} d(x_0 \otimes x_1 \otimes \ldots \otimes x_n) \; &:= \sum_{i<j} (-1)^{j+1} x_0 \otimes \ldots \otimes [x_i, x_j] \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_n \\ &\quad + [x_0, x_1] \otimes \ldots \otimes x_{n+1} + \sum_{i=1}^{n+1} (-1)^i [x_i, x_0] \otimes x_1 \otimes \ldots \otimes \hat{x}_i \otimes \ldots \otimes x_{n+1}, \end{aligned}$$

for $x_0 \in N$ and $x_1, \ldots, x_n \in \mathfrak{g}$. The homology of this new complex is called *Leibniz homology of* $\mathfrak{g}$ *with coefficients in* N, and denoted by $HL_*(\mathfrak{g}, N)$. The previous complex is recovered for any trivial corepresentation, for instance that on the ground field k, hence $HL_*(\mathfrak{g}) = HL_*(\mathfrak{g}, k)$.

Similarly, given a representation M, the cohomology theory dual to the homology $HL_*(\mathfrak{g}, N)$ is defined by the sequence of cochain modules and coboundary operators $\delta : \mathrm{Hom}(\mathfrak{g}^{\otimes n}, M) \longrightarrow \mathrm{Hom}(\mathfrak{g}^{\otimes n+1}, M)$, with

$$\begin{aligned}(\delta f)(x_1 \otimes \ldots \otimes x_{n+1}) \ &:= \sum_{i<j} (-1)^{j+1} f(x_1 \otimes \ldots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_{n+1}) \\ &+ [x_1, f(x_2 \otimes \ldots \otimes x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1 \otimes \ldots \otimes \hat{x}_i \otimes \ldots \otimes x_{n+1}), x_i]\end{aligned}$$

(remark that $[x_1, m]$ is not necessarily equal to $-[m, x_1]$ in the Leibniz representation M). It is called *Leibniz cohomology of* $\mathfrak{g}$ *with coefficients in* M, and denoted by $HL^*(\mathfrak{g}, M)$.

2.4 - Main results on Leibniz (co)homology. Among the main properties of Leibniz homology and cohomology, here are the ones, proved by J.-L. Loday and T. Pirashvili in [L2] and [LP], which are expected to hold for a natural homology theory on Leibniz algebras.

- On the free Leibniz algebra $\mathrm{Leib}(V)$ on a vector space V, only the first homology group is non-zero, that is, $HL_1(\mathrm{Leib}(V)) \cong V$ and $HL_n(\mathrm{Leib}(V)) = 0$ for any $n > 1$.
- *Leibniz homology and cohomology are derived functors* in the category of left modules over the universal enveloping algebra $UL(\mathfrak{g})$ of $\mathfrak{g}$. More precisely, if we denote by $\mathfrak{g}_{Lie}$ the Lie algebra obtained by identifying $[x, y]$ with $-[y, x]$, there are isomorphisms

$$\begin{aligned}HL_*(\mathfrak{g}, N) &\cong \mathrm{Tor}_*^{UL(\mathfrak{g})}(N, U(\mathfrak{g}_{Lie})), \\ HL^*(\mathfrak{g}, M) &\cong \mathrm{Ext}^*_{UL(\mathfrak{g})}(U(\mathfrak{g}_{Lie}), M).\end{aligned}$$

- *The second cohomology group of* $\mathfrak{g}$ *with coefficients in* M *classifies the abelian extensions of the Leibniz algebra* $\mathfrak{g}$ *by* M *up to equivalence*, that is,

$$HL^2(\mathfrak{g}, M) \cong \mathrm{Ext}(\mathfrak{g}, M).$$

- There is a *Künneth formula for Leibniz homology*: given two Leibniz algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$, there is a canonical isomoprphism

$$HL_*(\mathfrak{g}_1 \oplus \mathfrak{g}_2) \cong HL_*(\mathfrak{g}_1) \star HL_*(\mathfrak{g}_2),$$

  where the *free product of graded vector spaces* $\star$ satisfies $T(V \oplus W) \cong T(V) \star T(W)$, cf. [L3], [O].

2.5 - Leibniz invariants for Lie algebras. Let $\mathfrak{g}$ be a Lie algebra. In general, a representation M of a Lie algebra $\mathfrak{g}$ naturally determines a (symmetric) representation M and a (symmetric) corepresentation M^{op} of $\mathfrak{g}$ regarded as a Leibniz algebra. Then Lie and Leibniz (co)homology of $\mathfrak{g}$ with values in M are related by natural morphisms

$$\begin{aligned}HL_*(\mathfrak{g}, M^{op}) &\longrightarrow H_*^{Lie}(\mathfrak{g}, M), \\ H^*_{Lie}(\mathfrak{g}, M) &\longrightarrow HL^*(\mathfrak{g}, M).\end{aligned}$$

which are not necessarily injective nor surjective. Pirashvili showed in [P] that there exists a *relative homology theory* $H_*^{rel}(\mathfrak{g}, M)$ and an associated long exact sequence

$$\ldots \longrightarrow H_n^{rel}(\mathfrak{g}, M) \longrightarrow HL_n(\mathfrak{g}, M) \longrightarrow H_n^{Lie}(\mathfrak{g}, M) \longrightarrow H_{n-1}^{rel}(\mathfrak{g}, M) \longrightarrow \ldots$$

which allows to give more precise results for some specific Lie algebras.

- **Finite dimensional Lie algebras.** Let $\mathfrak{g}$ be a finite dimensional Lie algebra and M a right $\mathfrak{g}$-module.
  - if $H_*^{Lie}(\mathfrak{g}, M) = 0$ then $HL_*(\mathfrak{g}, M) = 0$.
  - if $\mathfrak{g}$ is a free Lie algebra, then

$$HL_n(\mathfrak{g}, M) = \begin{cases} H_1^{Lie}(\mathfrak{g}, \mathfrak{g})^{\otimes k} \otimes H_0^{Lie}(\mathfrak{g}, M), & \text{if } n = 2k \text{ is even,} \\ H_1^{Lie}(\mathfrak{g}, \mathfrak{g})^{\otimes k} \otimes H_1^{Lie}(\mathfrak{g}, M), & \text{if } n = 2k+1 \text{ is odd.} \end{cases}$$

  - if $\mathfrak{g}$ is semisimple, then $HL_n(\mathfrak{g}, M) = 0$ for any $n > 0$, cf. also [Nt].
  - some explicit computations have been done if $\mathfrak{g}$ is a two or three dimensional Lie algebra (nilpotent or not) over the complex numbers.

- **Leibniz version of Gelfand-Fuchs cohomology for the Lie algebras of vector fields.** Let M be a differentiable manifold and let $V(M) := \Gamma^\infty(TM)$ be the real Lie algebra of vector fields on M. Fix on $V(M)$ the usual C^∞-topology corresponding to uniform convergence of any derivative on compact sets of M. The *continuous* Lie cochain complex of $V(M)$ with coefficients in the real trivial representation, with the Chevalley-Eilenberg coboundary operator

$$\delta : \mathrm{Hom}_{cont}(\bigwedge^n V(M), \mathbf{R}) \longrightarrow \mathrm{Hom}_{cont}(\bigwedge^{n+1} V(M), \mathbf{R}),$$

  gives rise to the *Gelfand-Fuchs cohomology of* $V(M)$, which is denoted by $H^*_{GF}(V(M), \mathbf{R})$ or simply $H^*_{GF}(V(M))$. Many computations of Gelfand-Fuchs cohomology groups have been done, mainly in the 70's, by Gelfand and Fuchs, Bott and Segal, Losik, Guillemin, Trauber, and Haefliger, who related them to the invariants of some foliations on M, cf. [GF1], [GF2], [BS] and [H]. The main result is that the cohomology groups of vector fields on $\mathbf{R}^n$ form an exterior algebra with one generator in dimension $2n+1$, which is called the *Godbillon-Vey invariant.*

  Similarly, one can regard $V(M)$ as a continuous Leibniz algebra, and consider the submodules of continuous cochains $\mathrm{Hom}^{cont}(V(M)^{\otimes n}, \mathbf{R}) \subset \mathrm{Hom}(V(M)^{\otimes n}, \mathbf{R})$ with coefficients in the trivial representation. Since the Leibniz differential df of any continuous cochain f is still continuous, the sequence of continuous cochains forms a subcomplex of the Leibniz complex whose cohomology is called *continuous Leibniz cohomology of* $V(M)$ and still denoted by $HL^*(V(M))$.

  J. Lodder showed in [Lo] some similarities and some differences between Gelfand-Fuchs cohomology and continuous Leibniz cohomology. The main result for $\mathbf{R}^n$ is that the Godbillon-Vey invariant is still detected in dimension $2n+1$, and there is a new invariant in dimension $2n+2$. (These classes generate the Leibniz cohomology groups as a tensor algebra and not as an exterior algebra.)

- **Leibniz homology of topological spaces.** Let X be a topological space. If X is connected and simply connected, the rational homology $H_*(X, \mathbf{Q})$ can be computed as the Lie homology $H_*^{dgLie}(\lambda_\bullet(X))$, for a suitable differential graded Lie algebra $(\lambda_\bullet(X), d)$ functorially associated to the space X, which is called the *Quillen model of* X, cf. [Q]. Here, the symbol H_*^{dgLie} denotes a generalization of Lie homology which takes into account the differential graded structure, since the Chevalley-Eilenberg boundary operator is constructed using both the Lie bracket and the differential d given on $\lambda_\bullet(X)$.

The Quillen model of a space X can be very difficult to find, however, once $\lambda_\bullet(X)$ is given, its minimal model $L_\bullet(V)$ determines automatically the homology. In fact, the minimal model of $(\lambda_\bullet(X),d)$ is a free dg-Lie algebra $(L_\bullet(V),d')$ on a suitable graded vector space V, with a separable differential d' such that its dg-Lie homology recovers that of $(\lambda_\bullet(X),d)$. Hence, by definition of the Quillen algebra and of its minimal model we get a sequence of isomorphisms

$$H_{*+1}(X,\mathbf{Q}) \cong H_*^{dgLie}(\lambda_\bullet(X)) \cong H_*^{dgLie}(L_\bullet(V)) \cong V.$$

Of course, the dg-Lie homology H_*^{dgLie} has a Leibniz version HL_*^{dg}. We then define the *Leibniz homology of* X as the dg-Leibniz homology of the Quillen model of X, hence we have

$$HL_{*+1}(X,\mathbf{Q}) := HL_*^{dg}(\lambda_\bullet(X)) \cong HL_*^{dg}(L_\bullet(V)).$$

Remark that if we would consider a Leibniz version of the Quillen model we would achieve the leibniz homology of the *free Leibniz algebra on* V, which is again V, hence we would obtain no new invariants.

The example of the spheres $X = \mathbf{S}^n$, for $n \geq 2$, has been completely worked out by M. Livernet, cf. [Liv1], [Liv2]. The rational homology $H_*(\mathbf{S}^n,\mathbf{Q})$ of the sphere is zero everywhere except in degree n, where it is $\mathbf{Q}$. Hence the generator of the minimal model for the n-sphere is the graded vector space $V = \mathbf{Q}$ concentrated in degree n. The Leibniz homology $HL_*(\mathbf{S}^n,\mathbf{Q})$ then is equal to $\mathbf{Q}$ (i.e. it has one non-zero class) in each of the following dimensions:

$$\begin{array}{lll} \text{if } n \text{ is odd:} & n, 2n, 3n, \ldots & (\text{period } n) \\ \text{if } n \text{ is even:} & \left\{ \begin{array}{l} n, 4n-1, 5n-2, \ldots \\ 3n-1, 6n-2, \ldots \end{array} \right. & (\text{period } 3n-1). \end{array}$$

- **Rational homotopy of Leibniz spaces.** When X is a connected and simply connected topological space, its rational homotopy $\Pi_*(X,\mathbf{Q}) = \Pi_*(X)\otimes\mathbf{Q}$ is also completely determined by the Quillen model $\lambda_\bullet(X)$ and its minimal model $L_\bullet(V)$, cf. [Q].

  In fact, by costruction, the complexes $(\lambda_\bullet(X),d)$ and $(L_\bullet(V),d')$ are quasi-isomorphic, and there is a sequence of isomorphisms

  $$\Pi_{*+1}(X,\mathbf{Q}) \cong H_*(\lambda_\bullet(X),d) \cong H_*(L_\bullet(V),d'),$$

  where the last homology, for the uniqueness of the minimal model, only depends on the rationl homology $H_{*+1}(X,\mathbf{Q}) \cong V$.

  Remark that the rational homotopy of a space only depends on the differential graded structure of its Quillen model (and not on its Lie bracket), hence there exists no Leibniz version of it. What changes here is the category of spaces for which one can consider the rational homotopy: the dg-Lie algebra $\lambda_\bullet(X)$ can in fact be regarded as a trivial dg-Leibniz algebra. One can therefore define a *Leibniz (connected and simply connected) topological space* as (the virtual space functorially associated to) any dg-Leibniz algebra $(\lambda_\bullet,d)$: its *rational homology* is then $HL_{*+1}(\lambda_\bullet,\mathbf{Q}) := HL_*^{dg}(\lambda_\bullet)$ and its *rational homotopy* is $\Pi_{*+1}(\lambda_\bullet,\mathbf{Q}) := H_*(\lambda_\bullet,d)$.

  Do Leibniz spaces indeed exist? M. Livernet has found an example, the Leibniz spheres $\mathbf{S}\lambda^n$, cf. [Liv1], [Liv2]. For any $n \geq 2$, the *Leibniz sphere* $\mathbf{S}\lambda^n$ is the free dg-Leibniz algebra with one-dimensional rational homology concentrated in dimension n, that is,

$$HL_*(\mathbf{S}\lambda^n, \mathbf{Q}) = \begin{cases} \mathbf{Q}, & * = n \\ 0, & * \neq n \end{cases} .$$

Livernet showed that this dg-Leibniz algebra is unique up to isomorphism, and its rational homotopy is

$$\Pi_*(\mathbf{S}\lambda^n, \mathbf{Q}) = \begin{cases} \mathbf{Q}, & * = n, 2n-1, 3n-2, ... \quad (\text{period } n-1) \\ 0, & \text{elsewhere} \end{cases} .$$

3 Dialgebras

The computation of Leibniz homology in general bears the same difficulties as that of Lie homology. In the classical setting, some results are known when the Lie algebra is obtained from an associative algebra by means of the canonical bracket $[a, b] := a \cdot b - b \cdot a$. In particular, for the Lie algebra $\mathfrak{gl}_n(A)$ of matrices with coefficients in an associative algebra A, or better for the inductive limit $\mathfrak{gl}(A) = \mathfrak{gl}_\infty(A)$, the following fundamental theorem, due to Loday, Quillen and Tsygan, [LQ], [T], enables us to determine the Lie homology in terms of the cyclic homology of the algebra A (which is easier to compute): for any unital associative algebra A over a field of characteristic zero there is an isomorphism of graded Hopf algebras

$$H_*(\mathfrak{gl}(A)) \cong \bigwedge(HC_{*-1}(A)).$$

We then expect that Leibniz homology could be also computed in terms of some new homology theory. This lead J.-L. Loday to search for an 'associative version' of Leibniz algebras, together with its 'natural' and 'cyclic' homology theory.

3.1 - Dialgebras. A *dialgebra* is a vector space D equipped with two operations, $\dashv, \vdash \colon D \otimes D \longrightarrow D$, called left and right products, satisfying the following five axioms:

$$(D) \quad \begin{cases} a \dashv (b \dashv c) = (a \dashv b) \dashv c = a \dashv (b \vdash c), \\ (a \vdash b) \dashv c = a \vdash (b \dashv c), \\ (a \dashv b) \vdash c = a \vdash (b \vdash c) = (a \vdash b) \vdash c, \end{cases} \qquad a, b, c \in D.$$

Any associative algebra A is a trivial example of a dialgebra, for which the left and right products coincide, that is, $a \dashv b := a \cdot b =: a \vdash b$ for $a, b \in A$.

The motivation for considering the axioms (D) is that any dialgebra D determines uniquely a Leibniz algebra D_L, called *canonical Leibniz algebra of* D, whose bracket

$$[a, b] := a \dashv b - b \vdash a, \qquad a, b \in D,$$

is not antisymmetric, unless the left and right products coincide. (Observe that the choice of the axioms (D) is not unique.) The assignement $D \mapsto D_L$ yields a commutative diagram of categories and functors

$$\begin{tikzcd} \mathbf{Alg}_k \arrow[r, "-"] \arrow[d] & \mathbf{Lie}_k \arrow[d] \\ \mathbf{Dig}_k \arrow[r, "-"] & \mathbf{Leib}_k \end{tikzcd} .$$

Here are two non-trivial examples of dialgebras.

3.2 - The derived dialgebras of a differential associative algebra. To any (graded) associative differential algebra (A, d) we can give the structure of a dialgebra, by introducing the *derived products*

$$a \dashv b := a \cdot d(b), \qquad a \vdash b := d(a) \cdot b, \qquad a, b \in A$$

(with appropriate signs in the case of graded algebras), which satisfy the axioms (D) because $d(a\ db) = da\ db = d(da\ b)$. The triple $A_d := (A, \dashv, \vdash)$ is called *derived dialgebra* of (A, d). The canonical Leibniz algebra of the derived dialgebra A_d then coincides with the derived Leibniz algebra of the differential Lie algebra (A_L, d), as defined in (1.3), i.e. $(A_d)_L \cong (A_L)_d$.

In particular, the algebra of differential forms $(\Omega(M) = \oplus_{n\geq 0} \Omega^n(M), \wedge, d)$ on a differentiable manifold M can always be seen as a dialgebra, with left and right products

$$\omega \dashv \eta := \omega \wedge d(\eta) \qquad \text{and} \qquad \omega \vdash \eta := (-1)^{|\omega|+1} d(\omega) \wedge \eta.$$

Remark that if M is a Poisson manifold M, the Koszul bracket on the differential forms $\Omega(M)$ is *not* the Lie bracket associated to the exterior product of forms, and similarly the Schouten-Nijenhuis bracket on multivector fields $V(M) = \oplus_{n\geq 0} V^n(M)$ is *not* the Lie bracket associated to the exterior product. Instead, on differential forms one can define another Leibniz bracket

$$[\omega, \eta] = \omega \wedge d(\eta) + (-1)^{|\omega|} d(\omega) \wedge \eta,$$

which is not antisymmetric. (WHAT RELATION THEN???????????????)

3.3 - Dialgebra induced by a bimodule morphism. In [L4], Loday exhibits a typical example of a dialgebra, constructed on a bimodule M over an associative algebra A. If there is a morphism of A-bimodules $f : M \longrightarrow A$, then M is a dialgebra with products

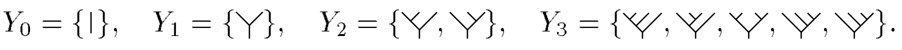
$$m \dashv m' := m \cdot f(m'), \qquad m \vdash m' := f(m) \cdot m', \qquad m, m' \in M.$$

In [F4] it is shown how this procedure can be inverted without restricting hypothesis, so all dialgebras can be constructed in this way.

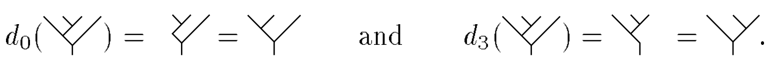
4 Dialgebra and symmetric (co)homology

The natural dialgebra boundary operator $d : D^{\otimes n} \longrightarrow D^{\otimes n-1}$ was introduced by Loday, on the hint that the composition $D^{\otimes 3} \longrightarrow D^{\otimes 2} \longrightarrow D^{\otimes 1}$ yields zero because of the relations (D). The chain module in dimension 3 must be 5 copies of $D^{\otimes 3}$, one for each equality of (D), and that in dimension 2 must be 2 copies of $D^{\otimes 2}$, one for each operation. In dimension n, it turns out to be $c_n = \frac{2n!}{n!(n+1)!}$ copies of $D^{\otimes n}$, where c_n is known as *Catalan number* and has many relevant combinatorical interpretations.

4.1 - The chain complex of planar binary trees. In order to define face maps on the chain modules of a dialgebra D, the best way to label the c_n copies of $D^{\otimes n}$ is to consider c_n as the cardinality of the set Y_n of *planar binary trees* with one root and $n+1$ leaves. Examples of such sets are:



The family of planar binary trees $\{Y_n\}$ is a pre-simplicial set with face maps $d_i : Y_n \longrightarrow Y_{n-1}$ which delete the i^{th} leaf of the trees, for any $i = 0, \dots, n$. For example,



For any field k, the boundary operator $d := \sum_{i=1}^{n-1}(-1)^i d_i$ satisfies $d \circ d = 0$, hence it defines a chain complex $d : k[Y_n] \longrightarrow k[Y_{n-1}]$.

4.2 - Dialgebra homology and cohomology. Let D be a dialgebra over a field k. The boundary operator defined on trees can be extended to a chain complex

$$d : k[Y_n] \otimes D^{\otimes n} \longrightarrow k[Y_{n-1}] \otimes D^{\otimes n-1}$$

by setting the face maps, for $i = 1, \ldots, n-1$, as

$$d_i(y \otimes (a_1 \otimes \ldots \otimes a_n)) := d_i(y) \otimes (a_1 \otimes \ldots \otimes a_i \circ_i^y a_{i+1} \otimes \ldots \otimes a_n),$$

where $\circ_i^y := \dashv$, if the i^{th} leaf of y is oriented as $\backslash$ and $\circ_i^y := \vdash$, if the i^{th} leaf of y is oriented as $/$. This operator still satisfy $d \circ d = 0$, because of the axioms (D). The homology of this complex is called *dialgebra homology of D with trivial coefficients*, and denoted by $HY_*(D, k)$ or $HY_*(D)$.

As for Leibniz algebras, for dialgebras it is not a priori clear which relations should be satisfied by the modules of coefficients for homology and cohomology theories. The good notions of *representations* and *corepresentation* of a dialgebra D is found with the same procedure sketched for Leibniz algebras, by introducing the *universal enveloping algebra* $E(D)$.

The dialgebra boundary operator can then be extended to a chain complex with coefficients in a corepresentation N of the dialgebra D. It suffices to define the extra face maps d_0, d_n that, for simplicity of the reader, we omit here (details can be found in [F4]). The homology of the complex

$$d : k[Y_n] \otimes N \otimes D^{\otimes n} \longrightarrow k[Y_{n-1}] \otimes N \otimes D^{\otimes n-1}$$

is called *dialgebra homology of D with coefficients in N*, and denoted by $HY_*(D, N)$.

Similarly, we can introduce a cochain complex

$$\delta : \mathrm{Hom}(k[Y_n] \otimes D^n, M) \longrightarrow \mathrm{Hom}(k[Y_{n+1}] \otimes D^{n+1}, M),$$

with coefficients in a representation M of D. The cohomology of this complex is called *dialgebra cohomology of D with coefficients in M*, and denoted by $HY^*(D, M)$. (Details on dialgebra cohomology can be found in [F4].)

4.3 - Main results on dialgebra (co)homology. The main properties of dialgebra homology and cohomology are proved in [L6] and [F4]:

- On the free dialgebra $\mathcal{D}ias(V)$ on a vector space V only the first homology group is non-zero, that is, $HY_1(\mathcal{D}ias(V)) \cong V$ and $HY_n(\mathcal{D}ias(V)) = 0$ for any $n > 1$.

- Dialgebra homology *is a generalization of Hochschild homology*, because

$$HY_n(A, M) \cong H_n(A, M),$$

  for any H-unital associative algebra A and any A-bimodule M.

- *The chain complex of bar-unital dialgebras with trivial coefficients is acyclic*, that is,

$$HY_n(D) = 0, \qquad n \geq 1.$$

  Here 'bar-unital' means that the dialgebra is equipped with a *bar-unit* (the best notion of unit which can be given to a dialgebra): an element 1 which is a unit only on the bar-side of the products, $a \dashv 1 = a = 1 \vdash a$ for any $a \in D$.

- *Dialgebra homology and cohomology are derived functors* in the category of left modules over the universal enveloping algebra $E(D)$ of a dialgebra D. More precisely,

$$HY_n(D,N) \cong \mathrm{Tor}_n^{E(D)}(N, M(D)),$$
$$HY^n(D,N) \cong \mathrm{Ext}^n_{E(D)}(M(D), M),$$

  where $M(D)$ is a suitable D-representation which coincide with the algebra itself when $D = A$ is an associative algebra.

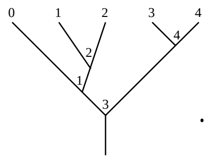
- *The second cohomology group classifies the abelian extensions of dialgebras up to equivalence*, that is,

$$HY^2(D,M) \cong \mathrm{Ext}(D,M),$$

  for any dialgebra D and any D-representation M.

So, dialgebra homology is the natural generalization of Hochschild homology to the dialgebra setting. However, following the classical philosophy, in order to compute the Leibniz homology of the matrices $\mathfrak{gl}(D)$, with coefficients in a dialgebra D, it is necessary to find also a generalization of cyclic homology of associative algebras.

4.4 - The chain complex of permutations. Let S_n be the set of permutations on n elements. This set is in bijection with the set $\tilde{Y}_n$ of *leveled planar binary trees* with one root and $n+1$ leaves. A tree is called *with level* if each of its n internal vertices is placed in one of the n levels of height (from the root to the leaves). For example, the tree is not leveled, while and are trees with levels. To describe the bijection $S_n \simeq \tilde{Y}_n$, it suffices to choose a rule to order the internal vertices of leveled trees, for instance, by giving to each vertex the same label as the leaf at the right hand side of the hole which is closed by the vertex, as in the example below:



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An n-tree with level is then identified with the permutation $\sigma \in S_n$ which associates to each labeled vertex its level. For instance, the tree of the example above is associated to the permutation $\binom{1\ 2\ 3\ 4}{2\ 3\ 1\ 4} = (123)$.

The face maps defined on planar binary trees can be easily extended to the trees with levels, and hence to the permutations: for all $i = 0, \ldots, n$ there exist face maps $d_i : S_n \longrightarrow S_{n-1}$ such that the operator $d = \sum_{i=0}^n (-1)^i d_i$ satisfies $d \circ d = 0$. Hence there is a chain complex $d : k[S_n] \longrightarrow k[S_{n-1}$.

4.5 - Symmetric homology. Let D be a dialgebra over a field k, and let N be a corepresentation of D. The boundary operator defined on permutations can be suitably extended to a chain complex

$$d : k[S_n] \otimes N \otimes D^{\otimes n} \longrightarrow k[S_{n-1}] \otimes N \otimes D^{\otimes n-1},$$

using the orientation of the leaves of the leveled tree associated to the permutation, which has the same form as the one given in (4.2). The new homology is called *symmetric homology of D with coefficients in N* and denoted by $HS_*(D,N)$. In particular, the symmetric homology of D with coefficients in itself is denoted by $HHS_*(D)$.

4.6 - Main results on symmetric homology. The symmetric homology is not proved to be a derived functor, nor to be the natural homology theory for a new kind of algebras. Its main application is the following theorem, proved in [F2] and [O]:

- *The Leibniz homology of dialgebras of infinite matrices is computed in terms of the symmetric homology.* More precisely, for any bar-unital dialgebra D there is an isomorphism of graded dual Leibniz algebras
$$HL_*(\mathfrak{gl}(D)) \cong T(HHS_{*-1}(D)).$$
- *The symmetric homology is also a generalization of Hochschild homology*, in fact for any unital associative algebra A and any unital bimodule M aver A we have
$$HS_n(A, M) \cong H_n(A, M).$$

4.7 - Conclusions. Dialgebra and symmetric homology are a generalization of Hochschild and cyclic homology respectively. They introduce some ...

- Representations and corepresentations are different. The universal enveloping algebras which classifies them is in general not isomorphic to its dual algebra, while for Lie and associative algebras it is.
- The duality between homology and cohomology theory is performed through the "cotangent complex" (chain complex with coefficients in the universal enveloping algebra) which is in general different from the bar complex, and not necessarily acyclic.
- The existence of a bar-unit in a dialgebra allows to define some degeneracy maps which satisfy all the simplicial relations except one. It is then natural to introduce two new categories of modules interpoling pre-simplicial and simplicial modules: that of pseudo-simplicial and that of almost-simplicial modules.

  The remarkable fact is that the objects $[n] := \{0, 1, ..., n\}$ of the simplicial category Δ can be interpretated as the sets of labeled leaves of rooted planar binary trees, and hence one can introduce the almost-simplicial category with objects given by the trees themselves belonging to Y_n, and the pseudo-simplicial category with objects given by the permutations belonging to S_n.

  These new categories are related to the cellular decomposition of the spheres given by the Stasheff's polyhedrons and the associahedrons respectively, but their geometric realization is for the moment completely unknown.

References

[BS] **Bott, R., Segal, G.** *The cohomology of vector fields on a manifold*, Topology **16** (1977), 285-298.

[F1] **Frabetti, A.** *Dialgebra homology of associative algebras*, C. R. Acad. Sci. Paris, **325** (1997), 135-140.

[F2] **Frabetti, A.** *Leibniz homology of dialgebras of matrices*, Journ. Pure Appl. Alg. **129** (1998), 123-141.

[F3] **Frabetti, A.** *Simplicial properties of the set of planar binary trees*, to appear in Journ. of Algebraic Combinatorics.

[F4] **Frabetti, A.** *Dialgebra (co)homology with coefficients*, to appear in a Springer Lecture Note.

[GF1] **Gelfand, M., Fuchs, D. B.** *Cohomologies of Lie algebra of tangential vector fields on a smooth manifold*, Funkt. Anal. i yevo prilozheniya **3.3** (1969), 32-54 and **4.2** (1970), 23-31.

[GF2] **Gelfand, M., Fuchs, D. B.** *Cohomologies of Lie algebra of vector fields with nontrivial coefficients*, Funkt. Anal. i yevo prilozheniya **4.3** (1970), 10-25.

[G] **Gnedbaye, A. V.** *Sur l'homologie des algèbres de Leibniz, Opérades des algèbres k-aires*, Thèse de l'Université Louis Pasteur (Strasbourg I), 1995.

[H] **Haefliger, A.** *Sur les classes caractéristiques des feuilletages*, Sém. Bourbaki **412** (1972).

[K] **Koszul, J.-L.** *Crochet de Schouten-Nijenhuis et cohomologie*, Soc. Math. France, Astérisque, hors série (1985), 257-271.

[K-S] **Kosmann-Schwarzbach, Y.** *From Poisson Algebras to Gerstenhaber Algebras*, Ann. Inst. Fourier **46** (1996), 1241-1272.

[Li] **Lichnerowicz, A.** *Les variétés de Poisson et leurs algèbres de Lie associées*, J. Diff. Geom. **12** (1977), 253-300.

[Liv1] **Livernet, M.** *Homotopie rationelle des algèbres de Leibniz*, C. R. Acad. Sci. Paris **325** (1997), 819-823.

[Liv2] **Livernet, M.** *Rational homotopy of Leibniz algebras*, Manuscripta Math. **96** (1998), 295-315.

[L1] **Loday, J.-L.** *Comparaison des homologies du groupe linéaire et de son algèbre de Lie*, Ann. Inst. Fourier **37**, 4 (1987), 167-190.

[L2] **Loday, J.-L.** *Une version non commutative des algèbres de Lie: les algèbres de Leibniz*, Ensegn. Math. **321** (1993), 269-293.

[L3] **Loday, J.-L.** *Künneth-style formula for the homology of Leibniz algebras*, Math. Zeit. **221** (1996) 41-47.

[L4] **Loday, J.-L.** *Algèbres ayant deux operations associatives (digèbres)*, C. R. Acad. Sci. Paris **321** (1995), 141-146.

[L5] **Loday, J.-L.** *Overview on Leibniz algebras, dialgebras and their homology*, Fields Inst. Commun. 17 (1997), 91–102.

[L6] **Loday, J.-L.** *Dialgebras*, to appear in a Springer Lecture Note.

[LP] **Loday, J.-L., Pirashvili, T.** *Universal Enveloping Algebras of Leibniz Algebras and (Co)homology*, Math. Annalen **296** (1993), 569-572.

[LQ] **Loday, J.-L., Quillen, D.** *Cyclic Homology and the Lie Algebra Homology of Matrices*, Comment. Math. Helv. **59** (1984), 565-591.

[Lo] **Lodder, J.** *Leibniz cohomology for differentiable manifolds*, Ann. Inst. Fourier, **48**, 1 (1998), 73-95.

[N] **Nijenhuis, A.** *Jacobi-type identities for bilinear differential concomitants of certain tensor fields, I*, Indag. Math. **17** (1955), 390-403.

[Nt] **Ntolo, P.** *Homologie de Leibniz d'algèbres de Lie semi-simples*, C. R. Acad. Sci. Paris **318** (1994), 707-710.

[O] **Oudom, J.-M.** *Cogèbres de Leibniz duales et homologie des algèbres de Leibinz*, Thèse de l'Université Montpellier II, 1997.

[P] **Pirashvili, T.** *On Leibniz homology*, Ann. Inst. Fourier, **44**, 2 (1994), 401-411.

[Q] **Quillen, D.** *Rational homotopy theory*, Ann. Math., **90**, 2 (1969), 205-295.

[S] **Schouten, J. A.** *On the differential operators of first order in tensor calculus*, Convegno Int. Geom. Diff. Italia, Ed. Cremonese, 1954.

[T] **Tsygan, B. L.** *The homology of matrix Lie algebras over rings and the Hochschild homologie* (in russian), Uspekhi Mat. Nauk **38** (1983), 217-218; Russ. Math. Survey **38** No. 2 (1983), 198-199.

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