Leibniz homology of dialgebras of matrices

Alessandra Frabetti

Dipartimento di Matematica - Università di Bologna, Italy

e-mail : frabetti@dm.unibo.it

April 28, 1997

Abstract

A dialgebra D is a vector space with two associative operations $\dashv, \vdash$ satisfying three more relations. By setting $[x,y] := x \dashv y - y \vdash x$, any dialgebra gives rise to a Leibniz algebra. Here we compute the Leibniz homology of the dialgebra of matrices $\mathfrak{gl}(D)$ with entries in a given dialgebra D. We show that $HL(\mathfrak{gl}(D))$ is isomorphic to the tensor module over $HHS(D)$, which is a variation of the natural dialgebra homology $HHY(D)$.

Let $(\mathfrak{g}, d)$ be a differential Lie algebra, with square zero differential d and Lie bracket $[\ ,\ ]_{\mathfrak{g}}$. The new bracket $[x,y] := [x, dy]_{\mathfrak{g}}$ which occurs in classical differential geometry (see e.g. [K] and the references in [KS]), is not antisymmetric, but satisfies the *Leibniz relation*

$(L)\quad [x,[y,z]] = [[x,y],z] - [[x,z],y].$

It is equivalent to the Jacobi relation when combined with antisymmetry, thus *Leibniz algebras* ([L1], [L-P]) are a generalization of Lie algebras.

A powerful method to study Lie algebras is to relate them with associative algebras. It is well known that an associative algebra gives rise to a Lie algebra, with bracket $[x,y] := xy - yx$. The same relation holds between Leibniz algebras and a new kind of objects called *dialgebras* (see [L2]). A dialgebra D is a vector space with two operations $\dashv, \vdash$ satisfying the following five axioms:

$$(D)\quad \begin{cases} a \dashv (b \dashv c) \overset{1}{=} (a \dashv b) \dashv c \overset{2}{=} a \dashv (b \vdash c) \\ (a \vdash b) \dashv c \overset{3}{=} a \vdash (b \dashv c) \\ (a \dashv b) \vdash c \overset{4}{=} a \vdash (b \vdash c) \overset{5}{=} (a \vdash b) \vdash c, \end{cases}$$

for any $x, y, z \in D$. In particular, associative algebras are dialgebras for which $\dashv\, = \,\vdash$. Any dialgebra gives rise to a Leibniz algebra, with bracket $[x,y] := x \dashv y - y \vdash x$.

A typical example of dialgebra is constructed from an associative differential algebra (A,d), where d is a square zero derivation on A, by setting $x \dashv y := x \cdot dy$ and $x \vdash y := dx \cdot y$. The Leibniz structure on (A,d) recovers the one defined on the differential Lie algebra $(\mathfrak{g}, d)$ when $\mathfrak{g}$ is the Lie algebra determined by A. This motivates the interest that dialgebras may arise in differential geometry and certain areas of theoretical physics.

A standard tool to construct invariants of algebras is homology theory. In particular, the relationship between the homology of the Lie algebra of matrices and the homology of the algebra of their entries is a classical problem. The main result, for unital associative algebras, is Loday-Quillen-Tsygan theorem ([L-Q], [T]), which relates the Lie homology of the inductive limit $\mathfrak{gl}(A) = \mathfrak{gl}_{r\to\infty}(A)$ to the cyclic homology of A: there is an isomorphism of graded Hopf algebras

$$H_*(\mathfrak{gl}(A), k) \cong \bigwedge(HC_{*-1}(A)).$$

For involutive algebras Loday and Procesi [L-Pr] have also computed the Lie homology of orthogonal matrices in terms of dihedral homology:

$$H_*(\mathfrak{o}(A), k) \cong \bigwedge(HD_{*-1}(A)),$$

and similarly for symplectic matrices.

On Leibniz algebras, which are not antisymmetric, the Lie homology theory is not appropriate. Leibniz algebras have a 'natural' homology theory, denoted by HL, where the exterior powers of the algebra, in the Chevalley-Eilenberg chain complex, are replaced by *tensor* powers. We can clearly compute Leibniz homology of Lie algebras. So, for unital associative algebras, Cuvier and Loday ([C], [L1]) have related the Leibniz homology of matrices to the Hochschild homology of the algebra of entries:

$$HL_*(\mathfrak{gl}(A), k) \cong T(HH_{*-1}(A))).$$

Since the matrices with entries in a dialgebra still form a dialgebra (and hence a Leibniz algebra), it is natural to try to compute the Leibniz homology of the dialgebra of matrices.

For dialgebras there exists a 'natural' homology theory, denoted by HY_*, where the chain modules are several copies of tensor powers of the dialgebra, one for each *rooted planar binary tree* ([L2]). This theory is one possible generalization of Hochschild homology to dialgebras. For instance, for an associative unital algebra A, there is an isomorphism $HHY_*(A) \cong HH_*(A)$ [F1]. (Recall that the Hochschild homology of A with coefficients in A is denoted by HH, so we adopt the same notation also for dialgebra homology with coefficients.)

A slight modification consists in replacing the planar binary trees by *permutations* (1.4). This gives a new homology theory for dialgebras, denoted by HS, which is closely related to Leibniz homology. In fact we prove the following

Theorem 2.2 *Let k be a characteristic zero field. Let D be a dialgebra over k equipped with a bar-unit (a unit just on the bar-side of the products $\dashv$, $\vdash$). Then*

$$HL_*(\mathfrak{gl}(D), k) \cong T(HHS_{*-1}(D)).$$

The relationship between the homology $HHS_*(D)$ and the dialgebra homology $HHY_*(D)$ is given by a canonical map (3.4)

$$HHS_*(D) \longrightarrow HHY_*(D).$$

This map is not an isomorphism, in general. However, when $D = A$ is an associative algebra with unit we can prove that these two homology groups are isomorphic ([C], [F1]).

Contents

1 Variation HHS of dialgebra homology.

1.1 - Dialgebras. **([L2])** Let k be a field with characteristic 0. A *dialgebra over* k is a k-vector space D with two linear operations, $\dashv, \vdash$: $D \otimes D \longrightarrow D$, called left and right products, satisfying the following five axioms:

$$(D) \quad \begin{cases} a \dashv (b \dashv c) \stackrel{1}{=} (a \dashv b) \dashv c \stackrel{2}{=} a \dashv (b \vdash c) \\ (a \vdash b) \dashv c \stackrel{3}{=} a \vdash (b \dashv c) \\ (a \dashv b) \vdash c \stackrel{4}{=} a \vdash (b \vdash c) \stackrel{5}{=} (a \vdash b) \vdash c, \end{cases}$$

for any $a, b, c \in D$. A *bar-unit* is an element $1 \in D$ which is a unit for the left and right products only on the bar-side, i.e. such that $1 \vdash a = a = a \dashv 1$, for any $a \in D$.

1.2 - Examples.

1. Any associative algebra is an example of dialgebra, with left and right products coinciding. Moreover one can show that if a dialgebra contains a bar-unit which is a unit also on the pointer-side of the products (i.e. such that $1 \dashv a = a = a \vdash 1$ for any a), then the left and right products coincide, so the dialgebra is in fact an associative unital algebra.

2. Another example of dialgebra is constructed from an associative differential algebra (A, d) as follows:

$$a \dashv b := a \cdot d(b)$$
$$a \vdash b := d(a) \cdot b,$$

   (with an appropriate sign in the case of graded algebras), for any $a, b \in A$.

3. The last example of dialgebra we mention is the k-vector space $\mathrm{M}_r(D)$ of matrices with entries in D, with 'row $\times$ column' products defined by

$$\begin{aligned} (\alpha \dashv \beta)_{ij} &:= \sum_{k=1}^{r} \alpha_{ik} \dashv \beta_{kj} \\ (\alpha \vdash \beta)_{ij} &:= \sum_{k=1}^{r} \alpha_{ik} \vdash \beta_{kj} \end{aligned}$$

   for any $\alpha, \beta \in \mathrm{M}_r(D)$.

More examples can be found in [L2].

1.3 - Presimplicial set of permutations. Let S_n denote the set of permutations on n elements. The family $S = \{S_n, n \geq 1\}$ is a presimplicial set (a simplicial set without degeneracies). The face maps $d_i : S_n \longrightarrow S_{n-1}$ are defined as follows.

For any permutation σ, let $d_i(\sigma)$ be the permutation on the numbers $1, ..., n-1$ obtained by removing the term $\max(\sigma(i), \sigma(i+1))$ from the image of σ, and rescaling the remaining numbers $\sigma(1)$, ..., $\widehat{\max(\sigma(i), \sigma(i+1))}$, ..., $\sigma(n)$ from 1 to $n-1$. (From now on we shall denote by $\overline{\sigma(1)}$, ..., $\widehat{\max(\sigma(i), \sigma(i+1))}$, ..., $\overline{\sigma(n)}$ any rescaled sequence.)

It is easy to verify the simplicial relations

$$d_i d_j = d_{j-1} d_i, \quad \text{for } i < j.$$

In fact on both sides the maps act on a permutation σ by removing the terms $\max(\sigma(i), \sigma(i+1))$ and $\max(\sigma(j), \sigma(j+1))$ in the two possible orders: first $\max(\sigma(j), \sigma(j+1))$ and then $\max(\sigma(i), \sigma(i+1))$, or the opposite.

1.4 - Definition. (Dialgebra homology $HHS_*(D)$.) Let D be a dialgebra over the field k. We denote by $HHS_*(D)$ the homology of the presimplicial module $CS_n(D) := k[S_n] \otimes D^{\otimes n+1}$, with face maps

$$d_i : k[S_n] \otimes D^{\otimes n+1} \longrightarrow k[S_{n-1}] \otimes D^{\otimes n}, \quad d_i(\sigma \otimes \underline{a}) = d_i(\sigma) \otimes d_i^\sigma(\underline{a}).$$

Here $d_i(\sigma)$ is the permutation defined in (1.3), and $d_i^\sigma(\underline{a})$ is given, for any $\underline{a} = (a_0, a_1, ..., a_n) \in D^{\otimes n+1}$, by

$$d_i^\sigma(a_0, a_1, ..., a_n) := \begin{cases} (a_0 \dashv a_1, ..., a_n), & \text{for } i = 0, \\ (a_0, ..., a_i \circ_i^\sigma a_{i+1}, ..., a_n), & \text{for } i = 1, ..., n-1, \\ (a_n \vdash a_0, ..., a_{n-1}), & \text{for } i = n, \end{cases}$$

with products $\circ_i^\sigma$ defined by

$$\circ_i^\sigma := \begin{cases} \dashv & \text{if } \sigma(i) < \sigma(i+1), \\ \vdash & \text{if } \sigma(i) > \sigma(i+1). \end{cases}$$

1.5 - Remark. The homology theory for a dialgebra D is well defined on coefficients being *corepresentations* of D. These are k-modules endowed with four actions of D satisfying 15 appropriate axioms, with the peculiar property that a dialgebra is *not* a corepresentation of itself. The 'homology of D' that we consider is, strictly talking, the homology with coefficients in a corepresentation D° defined on the k-module D, with actions depending - but not equal to! - on the left and right products of D. We call it anyway 'homology with coefficients in D', and we adopt for dialgebras the standard notation HH, recalling that for an associative algebra A the Hochschild homology with coefficients in A is denoted by $HH_*(A)$ instead of $H_*(A, A)$. (More details can be found in [F2].)

1.6 - Lemma. *The face maps defined above indeed satisfy the simplicial relations*

$$d_i d_j = d_{j-1} d_i, \qquad \textit{for } i < j.$$

Proof. After (1.3), we just need to show that for any $\sigma \in S_n$, and $\underline{a} = (a_0, a_1, ..., a_n) \in D^{\otimes n+1}$ the identity

$$d_i^{d_j(\sigma)} d_j^\sigma(\underline{a}) = d_{j-1}^{d_i(\sigma)} d_i^\sigma(\underline{a})$$

holds for any $i < j$.

Assuming that $i < j$, we consider separately the case $i+1 < j$ and the case $i+1 = j$. In the first case, namely $i+1 < j$, when $\min(\sigma(i), \sigma(i+1)) < \min(\sigma(j), \sigma(j+1))$ both maps $d_i^{d_j(\sigma)} d_j^\sigma$ and $d_{j-1}^{d_i(\sigma)} d_i^\sigma$ act on $\underline{a}$ as

$$(a_0, \ldots, a_i \circ_i^\sigma a_{i+1}, \ldots, a_j \circ_j^\sigma a_{j+1}, \ldots, a_n)$$

thus they are equal. Similarly, when $\min(\sigma(i), \sigma(i+1)) > \min(\sigma(j), \sigma(j+1))$ both maps $d_i^{d_j(\sigma)} d_j^\sigma$ and $d_{j-1}^{d_i(\sigma)} d_i^\sigma$ act on $\underline{a}$ as

$$(a_0, \ldots, a_j \circ_j^\sigma a_{j+1}, \ldots, a_i \circ_i^\sigma a_{i+1}, \ldots, a_n).$$

Thus again they are equal.

In the second case, namely $i+1 = j$, these two terms are overlapping one each other, and we can check the equality using the associative properties of the left and right products. In fact on one side we get:

$$d_i^{d_{i+1}(\sigma)} d_{i+1}^\sigma(\underline{a}) = (a_0, \ldots, a_i \circ_i^\sigma (a_{i+1} \circ_{i+1}^\sigma a_{i+2}), \ldots, a_n),$$

and on the other side:

$$d_i^{d_i(\sigma)} d_i^\sigma(\underline{a}) = (a_0, \ldots, (a_i \circ_i^\sigma a_{i+1}) \circ_{i+1}^\sigma a_{i+2}, \ldots, a_n).$$

So we just need to check the equality of the terms $a_i \circ_i^\sigma (a_{i+1} \circ_{i+1}^\sigma a_{i+2})$ and $(a_i \circ_i^\sigma a_{i+1}) \circ_{i+1}^\sigma a_{i+2}$ in the six cases shown by the following table:

Cases	$\sigma(i) < \sigma(i+1)$	$\sigma(i) > \sigma(i+1)$
$\sigma(i+1) < \sigma(i+2)$	$\sigma(i) < \sigma(i+1) < \sigma(i+2)$	$\sigma(i+1) < \sigma(i) < \sigma(i+2)$ $\sigma(i+1) < \sigma(i+2) < \sigma(i)$
$\sigma(i+1) > \sigma(i+2)$	$\sigma(i) < \sigma(i+2) < \sigma(i+1)$ $\sigma(i+2) < \sigma(i) < \sigma(i+1)$	$\sigma(i+2) < \sigma(i+1) < \sigma(i)$

It is easy now to check case by case:

- $\sigma(i) < \sigma(i+1) < \sigma(i+2) \Longrightarrow a_i \dashv (a_{i+1} \dashv a_{i+2}) = (a_i \dashv a_{i+1}) \dashv a_{i+2}$,
- $\sigma(i) < \sigma(i+2) < \sigma(i+1) \Longrightarrow a_i \dashv (a_{i+1} \vdash a_{i+2}) = (a_i \dashv a_{i+1}) \dashv a_{i+2}$,
- $\sigma(i+2) < \sigma(i) < \sigma(i+1) \Longrightarrow a_i \vdash (a_{i+1} \vdash a_{i+2}) = (a_i \dashv a_{i+1}) \vdash a_{i+2}$,
- $\sigma(i+1) < \sigma(i) < \sigma(i+2) \Longrightarrow a_i \vdash (a_{i+1} \dashv a_{i+2}) = (a_i \vdash a_{i+1}) \dashv a_{i+2}$,
- $\sigma(i+1) < \sigma(i+2) < \sigma(i) \Longrightarrow a_i \vdash (a_{i+1} \dashv a_{i+2}) = (a_i \vdash a_{i+1}) \dashv a_{i+2}$,
- $\sigma(i+2) < \sigma(i+1) < \sigma(i) \Longrightarrow a_i \vdash (a_{i+1} \dashv a_{i+2}) = (a_i \vdash a_{i+1}) \vdash a_{i+2}$.

Thus the six equalities are precisely the five axioms (D) of the dialgebra D (the 'central' axiom is used twice). □

1.7 - Theorem. *Let A be an associative unital algebra over k. Then*

$$HHS_*(A) \cong HH_*(A).$$

Proof. (Cuvier) It follows from the sequence of isomorphisms

$$HHS_*(A) \overset{2.14}{\cong} H_*(P(A)) \overset{[C]}{\cong} HH_*(A).$$

□

Alternative proof of (1.7). (Lodder) When A is an associative algebra, the products $\circ_i^\sigma$ coincide for all $\sigma \in S_n$ and $i = 0, ..., n$. The components d_i^σ of the face maps (see 1.4), then, do not depend on the permutation σ, thus their alternating sum $\sum_{i=0}^n (-1)^i d_i^\sigma$ is the Hochschild boundary on A. If A is endowed with a unit, the Hochschild complex $C_*(A, A)$ is endowed with the well known degeneracies

$$s_j(a_0, a_1, ..., a_n) := (a_0, ..., a_j, 1, a_{j+1}, ..., a_n), \quad \text{for } j = 0, ..., n,$$

which makes it into a simplicial module.

If the family $S = \{S_n, n \geq 1\}$ were a simplicial set, by the Eilenberg-Zilber theorem we knew that the complex $CS_*(A)$ was homotopy equivalent to the tensor product of the two complexes $k[S_*]$ and $C_*(A, A)$. Then, by the Künneth formula, we could argue that

$$HHS_n(A) = \oplus_{p+q=n} H_p(k[S_*]) \otimes H_q(C_*(A, A)) = HH_n(A),$$

since the complex $k[S_*]$ is obviously acyclic, i.e. $H_0(k[S_*]) = k$ and $H_p(k[S_*]) = 0$ for $p > 0$.

But S is *not* a simplicial set. J. Lodder shows in [Lo], lemma (2.6), that applying the Dold-Kan functor to $k[S]$ we obtain a simplicial module with the same homology, for which the previous discussion makes sense. A theoretical explaination of this result is given in [F1]: the family S is in fact a *pseudo*-simplicial set, i.e. it is endowed with degeneracy operators $s_j : S_n \longrightarrow S_{n+1}$, $j = 0, ..., n$, which commute, as usual, with the face maps,

$$d_i s_j = \begin{cases} s_{j-1} d_i & , i < j, \\ id & , i = j, j+1, \\ s_j d_{i+1} & , i > j+1, \end{cases}$$

but do *not* commute each other, i.e. they do not satisfy usual relations $s_i s_j = s_{j+1} s_i$ for $i \leq j$. Then the result follows from the fact that Eilenberg-Zilber theorem holds more generally for pseudo-simplicial modules. □

2 Leibniz homology of matrices of dialgebras.

Main theorem.

2.1 - Leibniz homology $HL_*(\mathfrak{gl}_r(D), k)$. Let D be a dialgebra over k, and $\mathrm{M}_r(D)$ the dialgebra of matrices with entries in D. We denote by $\mathfrak{gl}_r(D)$ the k-vector space $\mathrm{M}_r(D)$ equipped with the structure of Leibniz algebra given by the bracket

$$[\alpha, \beta] := \alpha \dashv \beta - \beta \vdash \alpha$$

for any $\alpha, \beta \in \mathrm{M}_r(D)$.

The *Leibniz homology* $HL_*(\mathfrak{gl}_r(D), k)$ of $\mathfrak{gl}_r(D)$ (with coefficients in the trivial module k) is the homology of the complex $(CL_*(\mathfrak{gl}_r(D), k), d)$:

$$\cdots \longrightarrow \mathfrak{gl}_r(D)^{\otimes n} \longrightarrow \mathfrak{gl}_r(D)^{\otimes n-1} \longrightarrow \cdots \longrightarrow \mathfrak{gl}_r(D) \longrightarrow k,$$

where the Leibniz differential $d : \mathfrak{gl}_r(D)^{\otimes n} \longrightarrow \mathfrak{gl}_r(D)^{\otimes n-1}$ is defined, for each $(\alpha_1, \ldots, \alpha_n) \in \mathfrak{gl}_r(D)^{\otimes n}$, by

$$d(\alpha_1, \ldots, \alpha_n) := \sum_{1 \leq i < j \leq n} (-1)^j (\alpha_1, \ldots, \alpha_{i-1}, [\alpha_i, \alpha_j], \ldots, \widehat{\alpha_j}, \ldots, \alpha_n) \tag{1}$$

and $d(\alpha) = 0$ for $\alpha \in \mathfrak{gl}_r(D)$. Notice that we adopt the notation $(\alpha_1, \ldots, \alpha_n) = \alpha_1 \otimes \cdots \otimes \alpha_n \in D^{\otimes n}$.

Our main result is stated in the next theorem. Let $\mathfrak{gl}(D) = \mathfrak{gl}_{r \to \infty}(D)$ denote the inductive limit of the inclusions $\mathfrak{gl}_r(D) \hookrightarrow \mathfrak{gl}_{r+1}(D)$.

2.2 - Theorem. *Let D be a dialgebra over k, endowed with bar-unit. There is an isomorphism of graded vector spaces*

$$HL_*(\mathfrak{gl}(D), k) \cong T(HHS_*(D)[1]),$$

where the standard notation $[1]$ means that the degree of the complex $CS_*(D)$ is shifted by 1, i.e. $CS_n(D)[1] = CS_{n-1}(D)$.

2.3 - Remark. B. Fresse recently proved in [Fr] that for any unital cooperad $\mathcal{P}$, if C is a connected graded $\mathcal{P}$-coalgebra equipped with a group structure, then C is a cofree graded $\mathcal{P}$-coalgebra. This means that $C = \bigwedge(\mathrm{Prim}C)$, when $\mathcal{P} = \mathcal{C}\wr\Updownarrow$, and $C = T(\mathrm{Prim}C)$ when $1\Box\,\mathcal{P}\Box = \mathcal{L}]\rangle\lfloor^!$.

Now, the $\mathcal{P}$-homology of matrices happens to be a group-like connected graded $\mathcal{P}^!$-coalgebra, thus it is predicted to be cofree, generated by its so called primitive elements. Applying the above result to $C = H_*^{\mathcal{L}\rangle]}(\mathfrak{gl}(A))$, we get $C = \bigwedge(\mathrm{Prim}C)$ (recall that $\mathcal{L}\rangle]^! = \mathcal{C}\wr\Updownarrow$). Loday-Quillen-Tsygan theorem states that $\mathrm{Prim}C = HC_{*-1}(A)$.

Similarly, for $C = H_*^{\mathcal{L}]\rangle\lfloor}(\mathfrak{gl}(D))$ (where $H^{\mathcal{L}]\rangle\lfloor} = HL$), we know that C must be of the form $T(\mathrm{Prim}C)$. Our result says more precisely that $\mathrm{Prim}C = HS_{*-1}(D)$.

Sketch of the proof. The idea of the proof is the one used by Cuvier and Loday for associative algebras, namely to find a complex $L_*(D)$ having the same homology as the Leibniz complex $CL_*(\mathfrak{gl}(D), k)$ but being small enough to be comparable, by means of the tensor functor T, to the complex $CS_*(D)$ defined in the previous section. The proof is organized in three main steps. First we find the complex $L_*(D)$ and show, in (2.5), that it has the same homology as $CL_*(\mathfrak{gl}(D), k)$. Then, in (2.13), we find the primitive part $P_*(D)$ of $L_*(D)$, and finally we show, in (2.14), that the complexes $P_*(D)$ and $CS_*(D)$ are isomorphic. □

2.4 - Remark. For an associative algebra A, J. Lodder [Lo] gave an alternative proof of Cuvier theorem. He introduces some multi-presimplicial modules $P^m(A)$, and then shows that they have a geometrical interpretation involving the James model of the classifying space BG of a discrete group, when $A = \mathbf{Q}[G]$ is its group ring.

His key observation is that each $P^m(A)$ is isomorphic to the *tensor product* of two multi-presimplicial modules $k[S^m(A)]$ and $C^m(A, A)$, which have well known homology. For dialgebras this is not anymore true, and his method brings no new insight. (Compare this situation with the one described in the alternative proof of theorem (1.7), where we already remarked that *associative* algebras bring additional fundamental properties to the dialgebra complexes.)

The complex $L_*(D)$.

Let S_{n+1} be the group of permutations on the $n+1$ elements $0, 1, \ldots, n$, and let U_{n+1} be the conjugacy class of the cycle $\tau = \tau_n = (012\cdots n) \in S_{n+1}$. For each $n \geq 0$, put $L_n(D) := k[S_{n+1}] \otimes D^{\otimes n+1}$, $P_n(D) := k[U_{n+1}] \otimes D^{\otimes n+1}$ and $C_n(D) := D^{\otimes n+1}$. We then have obvious inclusions of k-modules

$$\begin{array}{ccccc} C_n(D) & \hookrightarrow & P_n(D) & \hookrightarrow & L_n(D) \\ \underline{a} & \mapsto & \tau \otimes \underline{a} & & \\ & & \omega \otimes \underline{a} & \mapsto & \omega \otimes \underline{a}. \end{array}$$

2.5 - Proposition. *Let D be a dialgebra with bar-unit, then $L_*(D)$ can be made into a complex such that*

$$HL_*(\mathfrak{gl}(D), k) \cong H_*(L(D)[1]).$$

Proof. Denote by $\mathfrak{gl}_r$ the vector space $\mathfrak{gl}_r(k)$, and consider for each fixed r the natural structure of $\mathfrak{gl}_r$-module given on $\mathfrak{gl}_r(D)^{\otimes n}$ by the right action

$$[(\alpha_1, \ldots, \alpha_n), v] := \sum_{k=1}^{n} (\alpha_1, \ldots, \alpha_{k-1}, [\alpha_k, v], \ldots, \alpha_n)$$

where, for each $v \in \mathfrak{gl}_r$ and $\alpha_k \in \mathfrak{gl}_r(D)$ the right action

$$([\alpha_k, v])_{ij} := (\alpha_k \cdot v - v \cdot \alpha_k)_{ij} = \sum_l ((\alpha_k)_{il} \cdot v_{lj} - v_{il} \cdot (\alpha_k)_{lj})$$

is defined in terms of the k-right and left actions on D. Since D has a bar-unit 1, $\mathrm{M}_r(D)$ has also a bar-unit: the matrix $1 = (\delta_{ij})$. Lemma (2.6) permits us to embed $\mathfrak{gl}_r(k)$ into $\mathfrak{gl}_r(D)$, identifying each matrix $v \in \mathfrak{gl}_r$ with the matrix $v \cdot 1 = 1 \cdot v \in \mathfrak{gl}_r(D)$, in such a way that the $\mathfrak{gl}_r$-action on $\mathfrak{gl}_r(D)$ appears as a restriction of the adjoint action of the Leibniz algebra $\mathfrak{gl}_r(D)$.

The first step in the proof is based on the reductivity of $\mathfrak{gl}_r(k)$. This ensures that $\mathfrak{gl}_r(D)^{\otimes n}$ is completely reducible as a $\mathfrak{gl}_r$-module for any $n \geq 0$. So we can decompose $\mathfrak{gl}_r(D)^{\otimes n}$ into the direct sum

$$\mathfrak{gl}_r(D)^{\otimes n} = (\mathfrak{gl}_r(D)^{\otimes n})_{\mathfrak{gl}_r} \oplus Q_n,$$

where $(\mathfrak{gl}_r(D)^{\otimes n})_{\mathfrak{gl}_r}$ is the space of coinvariants of $\mathfrak{gl}_r(D)^{\otimes n}$, and Q_n is the direct sum of the remaining isotypical components in the isotypical decomposition of $\mathfrak{gl}_r(D)^n$ as a $\mathfrak{gl}_r$-module. The crucial point is that considering the space of coinvariants does not change the homology, as shown in (2.7).

We can then deduce that the family of projections

$$\mathfrak{gl}_r(D)^{\otimes n} \longrightarrow (\mathfrak{gl}_r(D)^{\otimes n})_{\mathfrak{gl}_r}, \quad \text{for } n \geq 0,$$

induces a quasi-isomorphism of complexes. This allows us to write the following isomorphism on homology,

$$HL_n(\mathfrak{gl}_r(D), k) := H_n(CL_*(\mathfrak{gl}_r(D), k), d) \cong H_n(CL_*(\mathfrak{gl}_r(D), k)_{\mathfrak{gl}_r}, \tilde{d})$$

where the last differential $\tilde{d}$ is obtained from the Leibniz differential d by passing to the quotient $\mathfrak{gl}_r(D)^{\otimes n} \longrightarrow (\mathfrak{gl}_r(D)^{\otimes n})_{\mathfrak{gl}_r}$.

The second step is just application of invariant theory, and repeats perfectly the Lie and Leibniz case for associative algebras. Let us recall briefly the sequence of isomorphisms on chains:

$$(\mathfrak{gl}_r(D)^{\otimes n})_{\mathfrak{gl}_r} \cong (\mathfrak{gl}_r{}^{\otimes n} \otimes D^{\otimes n})_{\mathfrak{gl}_r} \cong (\mathfrak{gl}_r{}^{\otimes n})_{\mathfrak{gl}_r} \otimes D^{\otimes n} \cong I_{n,r} \otimes D^{\otimes n},$$

where the ideal $I_{n,r}$ of $k[S_n]$ (whose definition can be found in [L1], (9.2.6)) coincides with $k[S_n]$ for $r \geq n$. Thus considering the inductive limit $\mathfrak{gl}(D)$ we have an isomorphism on homology

$$HL_*(\mathfrak{gl}(D), k) \cong H_*(k[S_n] \otimes D^{\otimes n}, \tilde{d}) = H_*(L(D)[1]),$$

and this proves the proposition. □

2.6 - Lemma. *Let D be a dialgebra with a bar-unit 1, then*

$$[\alpha, v] = [\alpha, v \cdot 1]$$

for each $\alpha \in \mathfrak{gl}_r(D)$ and $v \in \mathfrak{gl}_r$.

Proof. Notice that this equality is not trivial: on the left side the bracket is the action of $\mathfrak{gl}_r$ on $\mathfrak{gl}_r(D)$, while on the right side the bracket is the Leibniz bracket in $\mathfrak{gl}_r(D)$.

On the left side we have clearly

$$([\alpha, v])_{ij} = \sum_k (\alpha_{ik} \cdot v_{kj} - v_{ik} \cdot \alpha_{kj}).$$

On the right side we first notice that $v \cdot 1$ is a matrix of $\mathfrak{gl}_r(D)$ with entries

$$(v \cdot 1)_{ij} = \sum_l v_{il} \cdot \delta_{lj} = v_{ij} \cdot 1,$$

therefore we have:

$$\begin{aligned} ([\alpha, v \cdot 1])_{ij} &= \sum_k [\alpha_{ik} \dashv (v \cdot 1)_{kj} - (v \cdot 1)_{ik} \vdash \alpha_{kj}] \\ &= \sum_k (\alpha_{ik} \dashv v_{kj} \cdot 1 - v_{ik} \cdot 1 \vdash \alpha_{kj}) \\ &= \sum_k (\alpha_{ik} \cdot v_{kj} \dashv 1 - 1 \vdash v_{ik} \cdot \alpha_{kj}) \\ &= \sum_k (\alpha_{ik} \cdot v_{kj} - v_{ik} \cdot \alpha_{kj}), \end{aligned}$$

and the equality is proved. □

2.7 - Lemma. *For any $n \geq 0$ the action of $\mathfrak{gl}_r$ on $\mathfrak{gl}_r(D)^{\otimes n}$ commutes with the Leibniz differential on $\mathfrak{gl}_r(D)^{\otimes n}$, passing to the quotient into the homology. The induced action of $\mathfrak{gl}_r$ on the Leibniz homology $HL_n(\mathfrak{gl}_r(D))$ is trivial.*

Proof. After lemma (2.6), the compatibility of the Leibniz differential with the adjoint action of $\mathfrak{gl}_r(D)$, and thus of $\mathfrak{gl}_r$, on $\mathfrak{gl}_r(D)^{\otimes n}$, namely that

$$d[\alpha, v] = [d\alpha, v]$$

for each $\alpha \in \mathfrak{gl}_r(D)^{\otimes n}$ and $v \in \mathfrak{gl}_r$, is well known (see (10.6.3.0) in [L1]). To prove the second statement we define, for each $v \in \mathfrak{gl}_r$, the map $h(v) : \mathfrak{gl}(D)^{\otimes n} \longrightarrow \mathfrak{gl}(D)^{\otimes n+1}$ as

$$h(v)(\alpha) \ := \ (-1)^{n-1}\alpha \otimes (v \cdot 1).$$

It is easy to verify that the identity

$$d(h(v)(\alpha)) + h(v)(d\alpha) = [\alpha, v]$$

holds for any $\alpha \in \mathfrak{gl}_r(D)^{\otimes n}$. Thus the map $h(v)$ is a homotopy from $[-, v]$ to 0. In other words the action $[-, v]$ is trivial on the homology. □

The complex of primitives, $P_*(D)$.

In order to determine the primitive part of the complex $L_*(D)$, we need an explicit description of its differential. First of all we introduce some notations on permutations.

2.8 - Notations. Let σ be a permutation on the $n+1$ elements $\{0, 1, ..., n\}$. For each index $i \in \{0, ..., n\}$, we define permutations $\sigma^{(\pm i)}$ on n elements in the following way.

- For each index i such that $i < \sigma(i)$, let $\sigma^{(+i)}$ be the permutation on the n elements $\{0, 1, ..., \widehat{\sigma(i)}, ..., n\}$ which sends $j \neq i$ to $\sigma(j)$, and i to $\sigma^2(i)$.
- For each index i such that $i < \sigma^{-1}(i)$, let $\sigma^{(-i)}$ be the permutation on the n elements $\{0, 1, ..., \widehat{\sigma^{-1}(i)}, ..., n\}$ which assigns the values of σ (clearly missing the value i).

Finally we rescale all numbers from 0 to $n-1$.

For example from the cycle $\sigma = (0231)$ we obtain:

$$\begin{aligned}
0 < \sigma(0) = 2 \quad &\Longrightarrow \quad \sigma^{(+0)} = (031) \text{ on } \{0,1,3\} \\
&\Longrightarrow \quad \sigma^{(+0)} = (021) \text{ on } \{0,1,2\} \\
0 < \sigma^{-1}(0) = 1 \quad &\Longrightarrow \quad \sigma^{(-0)} = (023) \text{ on } \{0,2,3\} \\
&\Longrightarrow \quad \sigma^{(-0)} = (012) \text{ on } \{0,1,2\} \\
1 > \sigma(1) = 0 \quad &\Longrightarrow \quad \sigma^{(+1)} \text{ is not defined} \\
1 < \sigma^{-1}(1) = 3 \quad &\Longrightarrow \quad \sigma^{(-1)} = (021) \text{ on } \{0,1,2\} \\
2 < \sigma(2) = 3 \quad &\Longrightarrow \quad \sigma^{(+2)} = (021) \text{ on } \{0,1,2\} \\
2 > \sigma^{-1}(2) = 0 \quad &\Longrightarrow \quad \sigma^{(-2)} \text{ is not defined} \\
3 > \sigma(3) = 1 \quad &\Longrightarrow \quad \sigma^{(+3)} \text{ is not defined} \\
3 > \sigma^{-1}(3) = 2 \quad &\Longrightarrow \quad \sigma^{(-3)} \text{ is not defined.}
\end{aligned}$$

2.9 - Lemma. (The complex $L_*(D)$.) *The complex $L_*(D)$ has differential $\tilde{d} : k[S_{n+1}] \otimes D^{\otimes n+1} \longrightarrow k[S_n] \otimes D^n$ given by:*

$$\tilde{d}(\sigma \otimes (a_0, \dots, a_n)) = \sum_{0 \leq i < \sigma(i) \leq n} (-1)^{\sigma(i)+1}\, \sigma^{(+i)} \otimes (a_0, \dots, a_i \dashv a_{\sigma(i)}, \dots, \widehat{a_{\sigma(i)}}, \dots, a_n)$$
$$- \sum_{0 \leq i < \sigma^{-1}(i) \leq n} (-1)^{\sigma^{-1}(i)+1}\, \sigma^{(-i)} \otimes (a_0, \dots, a_{\sigma^{-1}(i)} \vdash a_i, \dots, \widehat{a_{\sigma^{-1}(i)}}, \dots, a_n).$$

Proof. The differential $\tilde{d}$ on the complex $L_*(D)$ can be explicitely found by applying the isomorphisms described in the proof of proposition (2.5). More precisely, for each $\sigma \otimes (a_0, \dots, a_n) \in k[S_{n+1}] \otimes D^{\otimes n+1}$, let $E^{a_0}_{0\sigma(0)} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}$ be a representative element in $\mathfrak{gl}(D)^{\otimes n+1}$ of its image class in $(\mathfrak{gl}(D)^{\otimes n+1})_{\mathfrak{gl}_r}$ under the isomorphism described above. Let us compute its Leibniz differential, but remark that, with the shifted numbering, the element $E^{a_i}_{i\sigma(i)}$ now corresponds to the element α_{i+1} of formula (1). We have

$$\begin{aligned}
\tilde{d}(\sigma \otimes (a_0, \dots, a_n)) &:= d(E^{a_0}_{0\sigma(0)} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}) \\
&= \sum_{0 \leq i < j \leq n} (-1)^{j+1} E^{a_0}_{0\sigma(0)} \otimes \cdots \otimes [E^{a_i}_{i\sigma(i)}, E^{a_j}_{j\sigma(j)}] \otimes \cdots \otimes \widehat{E^{a_j}_{j\sigma(j)}} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}) \\
&= \sum_{0 \leq i < j \leq n} (-1)^{j+1} E^{a_0}_{0\sigma(0)} \otimes \cdots \otimes E^{a_i}_{i\sigma(i)} \dashv E^{a_j}_{j\sigma(j)} \otimes \cdots \otimes \widehat{E^{a_j}_{j\sigma(j)}} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}) \\
&- \sum_{0 \leq i < j \leq n} (-1)^{j+1} E^{a_0}_{0\sigma(0)} \otimes \cdots \otimes E^{a_j}_{j\sigma(j)} \vdash E^{a_i}_{i\sigma(i)} \otimes \cdots \otimes \widehat{E^{a_j}_{j\sigma(j)}} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}) \\
&= \sum_{0 \leq i < \sigma(i) \leq n} (-1)^{\sigma(i)+1} E^{a_0}_{0\sigma(0)} \otimes \cdots \otimes E^{a_i \dashv a_{\sigma(i)}}_{i\sigma^2(i)} \otimes \cdots \otimes \widehat{E^{a_j}_{\sigma(i)\sigma^2(i)}} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}) \\
&- \sum_{0 \leq i < \sigma^{-1}(i) \leq n} (-1)^{\sigma^{-1}(i)+1} E^{a_1}_{0\sigma(0)} \otimes \cdots \otimes E^{a_{\sigma^{-1}(i)} \vdash a_i}_{\sigma^{-1}(i)\sigma(i)} \otimes \cdots \otimes \widehat{E^{a_{\sigma^{-1}(i)}}_{\sigma^{-1}(i)i}} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}).
\end{aligned}$$

Then taking the class of the elements E^a_{ij}, and using the permutations given in (2.8), we obtain the final formula given above. □

2.10 - Lemma. *For each $n \geq 0$ the inclusions of modules $C_n(D) \hookrightarrow P_n(D) \hookrightarrow L_n(D)$ commute with the differential $\tilde{d}$ on $L_n(D)$, therefore $P_*(D)$ and $C_*(D)$ are sub-complexes of $L_*(D)$, with differentials $\tilde{b}$ given by restricting $\tilde{d}$.*

Proof. We have to show first that for each $\omega \otimes \underline{a} \in P_n(D)$, i.e. $\omega = \sigma^{-1}\tau_n\sigma \in U_{n+1}$ with $\sigma \in S_{n+1}$ and $\tau_n = (01 \cdots n)$, the element $\tilde{d}(\omega \otimes \underline{a})$ is contained in $P_{n-1}(D)$, and secondly that when $\omega = \tau_n$ then $\tilde{d}(\tau_n \otimes \underline{a}) = \tau_{n-1} \otimes \underline{a}'$.

First notice that U_{n+1} is in fact the subset of S_{n+1} containing all the cycles of maximal length $n+1$, thus the permutation of an element $\omega \otimes \underline{a}$ in $P_n(D) = k[U_{n+1}] \otimes D^{\otimes n+1}$ will be written as the cycle $\omega = (\omega_0\omega_1 \cdots \omega_n)$. Assuming that $\omega_0 = 0$, we also know that $\omega = \sigma^{-1}\tau_n\sigma$, where the setting $\sigma(i) := \omega_i$, for $i = 0, 1, \dots, n$, defines a permutation $\sigma \in S_{n+1}$ which in fact belongs to S_n, because $\sigma(0) = \omega_0 = 0$. With this notation the differential $\tilde{d}$, described in lemma (2.9), on $\omega \otimes \underline{a}$ becomes

$$\tilde{d}(\omega \otimes \underline{a}) = \sum_{0 \leq \omega_i < \omega_j \leq n} (-1)^{\omega_j+1} E^{a_{\omega_0}}_{\omega_0\omega_1} \otimes \cdots \otimes [E^{a_{\omega_i}}_{\omega_i\omega_{i+1}}, E^{a_{\omega_j}}_{\omega_j\omega_{j+1}}] \otimes \cdots \otimes \widehat{E^{a_{\omega_j}}_{\omega_j\omega_{j+1}}} \otimes \cdots \otimes E^{a_n}_{n\sigma(n)}.$$

The summands are different from zero only if they contain either the commutator

$$[E^{a_{\omega_i}}_{\omega_i\omega_{i+1}}, E^{a_{\omega_{i+1}}}_{\omega_{i+1}\omega_{i+2}}] = E^{a_{\omega_i} \dashv a_{\omega_{i+1}}}_{\omega_i\omega_{i+2}} \tag{2}$$

or

$$[E^{a_{\omega_{i+1}}}_{\omega_{i+1}\omega_{i+2}}, E^{a_{\omega_i}}_{\omega_i\omega_{i+1}}] = -E^{a_{\omega_i} \vdash a_{\omega_{i+1}}}_{\omega_i\omega_{i+2}}, \tag{3}$$

which is placed at the $\min(w_i, w_{i+1})$-position. Thus $\tilde{d}(\omega\otimes\underline{a})$ is of the form

$$\sum_{0\leq\omega_i<\omega_j\leq n}(-1)^{\omega_j+1}E^{a_{\omega_0}}_{\omega_0\omega_1}\otimes\cdots\otimes\pm E^{a_{\omega_i}\star a_{\omega_{i+1}}}_{\omega_i\omega_{i+2}}\otimes\cdots\otimes E^{a_n}_{n\sigma(n)},$$

where $\star$ means $\dashv$ or $\vdash$. Therefore, written as a sum of elements of $k[S_n]\otimes D^{\otimes n}$, $\tilde{d}(\omega\otimes\underline{a})$ contains permutations obtained from the cycle ω of length $n+1$ by deleting the term $\min(\omega_i,\omega_{i+1})$, which are in fact cycles of length n. This proves that $\tilde{d}(\omega\otimes\underline{a})\in k[U_n]\otimes D^{\otimes n}=P_{n-1}(D)$.

In particular, when $\omega=\tau_n$, we have $\omega_i=i$ for each $i=0,1,\ldots,n$. Thus $\omega_i=i<i+1=\omega_{i+1}$ for each $i=0,1,\ldots,n-1$, and $\omega_n=n>0=\omega_{n+1}$. Then $\tilde{d}(\tau_n\otimes\underline{a})$ contains the cycles $(01\cdots\widehat{i+1}\cdots n)$, for $i=0,\ldots,n-1$, and $(01\cdots n-1)$, which are all τ_{n-1} after having rescaled the numbers. This proves that $\tilde{d}(\tau\otimes\underline{a})\in k\cdot\tau\otimes D^{\otimes n}=C_{n-1}(D)$. □

2.11 - Corollary. (The complex $P_*(D)$.) *The complex $P_*(D)$ is presimplicial, with the following faces $\tilde{d}_i : k[U_{n+1}]\otimes D^{\otimes n+1}\longrightarrow k[U_n]\otimes D^n$. For any $\omega\otimes\underline{a}\in P_n(D)$,*

$$\tilde{d}_i(\omega\otimes\underline{a})=\mathrm{sgn}(i,\omega)\ d_i(\omega)\otimes d^\omega_i(\underline{a}),$$

where

- $\mathrm{sgn}(i,\omega)=(-1)^i\cdot(-1)^{\max(\omega_i,\omega_{i+1}+1)}\cdot\varepsilon^\omega_i$, *with*

$$\varepsilon^\omega_i:=\begin{cases}+ & ,\ \textit{if } \omega_i<\omega_{i+1}\\ - & ,\ \textit{if } \omega_i>\omega_{i+1}\end{cases}$$

- *the cycle $d_i(\omega)=(\overline{\omega}_0\overline{\omega}_1\cdots\overline{\omega}_{n-1})$ is obtained by deleting the term $\max(\omega_i,\omega_{i+1})$ and rescaling the numbers from 0 to $n-1$:*

$$\overline{\omega}_k=\begin{cases}\omega_k & ,\ \textit{if } \omega_k<\max(\omega_i,\omega_{i+1})\\ \omega_k-1 & ,\ \textit{if } \omega_k>\max(\omega_i,\omega_{i+1})\end{cases}$$

- *the element $d^\omega_i(\underline{a})$ is obtained by substituting, at the $\min(\omega_i,\omega_{i+1})$-place, the term $a_{\min(\omega_i,\omega_{i+1})}$ with the term $a_{\omega_i}\circ^\omega_i a_{\omega_{i+1}}$, where $\circ_i : U_{n+1}\longrightarrow\{\dashv,\vdash\}$ is the map defined by*

$$\circ^\omega_i:=\begin{cases}\dashv, & \textit{if } \omega_i<\omega_{i+1},\\ \vdash, & \textit{if } \omega_i>\omega_{i+1}.\end{cases}$$

and deleting, at the $\max(\omega_i,\omega_{i+1})$-place, the term $a_{\max(\omega_i,\omega_{i+1})}$. In conclusion:

$$d^\omega_i(\underline{a})=(a_0,...,a_{\omega_i}\circ^\omega_i a_{\omega_{i+1}},...,a_n).$$

Proof. The differential $\tilde{b}$ on $P_*(D)$ is described in lemma (2.10). It is clear that $\tilde{b}(\omega\otimes\underline{a})$ can be written as a sum indexed by $i=0,...,n$. The addend corresponding to a given i contains one of the two commutators (2) or (3), according to which case, $\omega_i<\omega_{i+1}$ or $\omega_{i+1}<\omega_i$, happens to hold for i. Then it is easy to check that these addends are precisely $(-1)^i$ times the face maps given above. □

2.12 - Remark.(The complex $C_*(D)$.) The complex $C_*(D)$ is also presimplicial. Its faces are

$$\tilde{d}_i(\underline{a})=\begin{cases}(a_0,\ldots,a_i\dashv a_{i+1},\ldots,a_n) & ,\ \text{for } i=0,...,n-1,\\ (a_n\vdash a_0,a_1,\ldots,a_{n-1})1 & ,\ \text{for } i=n.\end{cases}$$

Thus when the two products $\dashv$ and $\vdash$ coincide, and D is an associative algebra A, the complex $(C_*(A), \tilde{b})$ is the Hochschild complex of A with coefficients in A.

Cuvier showed that if A has a unit the inclusion $C_*(A) \hookrightarrow P_*(A)$ is a quasi-isomorphism of complexes. He constructs an explicit chain homotopy, using an *extra-degeneracy* on the complex $P_*(A)$, which needs the existence of the unit in A. Thus his proof cannot be extended to non-unital algebras, nor to dialgebras even with bar-unit.

An alternative proof given by Lodder has already been discussed for theorem (1.7) above. It has the advantage of putting in evidence the two key assumptions which motivate the result:

- the presimplicial (in fact pseudo-simplicial) set of permutations S_* is *acyclic*,
- the complex $C_*(A)$ is a *simplicial* module.

The second key assumption is clearly missing for non-unital algebras and bar-unital dialgebras.

However, one can give a detailed description of the relation between the complex $C_*(D)$ and the dialgebra complex $CY_*(D)$, which will be related to $P_*(D)$ in section 3 below. In [F3] it is shown in full generality that $CY_*(D)$ is the total complex of a bicomplex where $C_*(D)$ is the first column. Then, as a contra-example to the above isomorphism for non-unital associative algebras, it is shown that for the free non-unital algebra $\overline{T}(V)$ we have

$$HHY_2(\overline{T}(V)) = V^{\otimes 3}, \quad \text{while} \quad HH_2(\overline{T}(V)) = 0.$$

2.13 - Proposition. *Let D be a dialgebra, and $L_*(D)$, $P_*(D)$ be the complexes defined above. Then*

$$L_*(D)[1] \cong T(P_*(D)[1]).$$

Proof. The isomorphism above does not explicitly involve the dialgebra structure of D. Thus we can simply adapt the proof of the analogue isomorphism $L_*(A)[1] \cong T(C_*(A)[1])$ given in [L1], (10.6.16) to the complex $P_*(D)$. In other words, we only need to verify that Lodder' multi-presimplicial construction $L_{*-1}(A) \cong \sum_{m\geq 1} P^m_{*-1}(A)$ still makes sense in the dialgebras setting.

At the chain level, the isomorphism $L_{n-1}(D) \cong T_{n-1}(P(D))$ clearly follows from the isomorphism of k-modules $k[S_n] \cong T_n(k[U_*])$, which means that any permutation $\sigma \in S_n$ can be written as a product of m cycles $\omega^{(1)}, \ldots, \omega^{(m)}$ of different lengths $n_1, \ldots, n_m$.

For the differentials, the following formula

$$T(\tilde{b})(\sigma \otimes \underline{a}) = \sum_{k=1}^{m} (-1)^{N_{k-1}} (\omega^{(1)} \otimes \underline{a}^{(1)}) \otimes \cdots \otimes \tilde{b}(\omega^{(k)} \otimes \underline{a}^{(k)}) \otimes \cdots \otimes (\omega^{(m)} \otimes \underline{a}^{(m)})$$

shows that $T(\tilde{b})$ does not depend on the explicit action of $\tilde{b}$. Since $\tilde{b}$ on $P_*(D)$ is defined as restriction of $\tilde{d}$ on $L_*(D)$, it is clear that the differentials $\tilde{d}$ and $T(\tilde{b})$ correspond under the chain isomorphism. $\square$

Isomorphism between $P_*(D)$ and $CS_*(D)$.

For any natural number n, consider the left action of S_n on $D^{\otimes n+1}$ given by

$$\sigma^{-1} \cdot (a_0, a_1, ..., a_n) = (a_0, a_{\sigma(1)}, ..., a_{\sigma(n)}).$$

If $\omega = (\omega_0, \omega_1, ..., \omega_n) \in U_{n+1}$ is a cycle with $\omega_0 = 0$, and $\sigma \in S_n$ is the permutation $\sigma(i) := \omega_i$, for $i = 1, ..., n$, which corresponds uniquely to ω, then the action of S_n on $D^{\otimes n+1}$ reads

$$\sigma^{-1} \cdot (a_0, a_1, ..., a_n) = (a_{\omega_0}, a_{\omega_1}, ..., a_{\omega_n}).$$

Let D be a dialgebra, and $CS_*(D)$, $P_*(D)$ be the above complexes. Define a map

$$\varphi : P_*(D) \longrightarrow CS_*(D)$$

for any $n \geq 0$ and $\omega \otimes \underline{a} \in k[U_{n+1}] \otimes D^{\otimes n+1}$, by

$$\varphi(\omega \otimes \underline{a}) := \mathrm{sgn}(\sigma) \cdot \sigma \otimes \sigma^{-1} \cdot \underline{a},$$

where $\sigma \in S_n$ is the permutation which corresponds to the cycle ω.

2.14 - Proposition. *Let D be a dialgebra. Then $\varphi : P_*(D) \longrightarrow CS_*(D)$ is an isomorphism of complexes. Therefore*

$$H_*(P(D)) \cong HHS_*(D).$$

Proof. Since φ is obviously an isomorphism of k-modules for any $n \geq 0$, we just need to show that it commutes with the differentials, i.e. that $d\varphi = \varphi\tilde{d}$. We shall show more, namely that $d_i\varphi = \varphi\tilde{d}_i$ holds for any $i = 0, ..., n$.

Let $\omega \otimes \underline{a}$ belong to $k[U_{n+1}] \otimes D^{\otimes n+1}$. On one side we have:

$$\begin{aligned} d_i\varphi(\omega \otimes \underline{a}) &= \mathrm{sgn}(\sigma) \cdot d_i(\sigma \otimes \sigma^{-1} \cdot \underline{a}) \\ &= \mathrm{sgn}(\sigma) \cdot d_i(\sigma) \otimes d_i^\sigma(\sigma^{-1} \cdot \underline{a}), \end{aligned}$$

and on the other side:

$$\begin{aligned} \varphi\tilde{d}_i(\omega \otimes \underline{a}) &= \mathrm{sgn}(\omega, i) \cdot \varphi(d_i(\omega) \otimes \tilde{d}_i^\omega(\underline{a})) \\ &= \mathrm{sgn}(\omega, i)\mathrm{sgn}(d_i\omega) \cdot \sigma' \otimes (\sigma')^{-1} \cdot \tilde{d}_i^\omega(\underline{a}), \end{aligned}$$

where $\sigma' = d_i(\sigma)$ is the permutation associated to the cycle $d_i(\omega)$. We only need to check that

$$\mathrm{sgn}(\sigma) = \mathrm{sgn}(\omega, i) \cdot \mathrm{sgn}(d_i\omega)$$

and

$$d_i^\sigma(\sigma^{-1} \cdot \underline{a}) = (\sigma')^{-1} \cdot \tilde{d}_i^\omega(\underline{a}).$$

Since the first equality does not depend on the dialgebra stucture, and has been shown by Lodder in [Lo], lemma (2.4), we only need to check the second.

Both sides consist of two steps, performed in the two possible orders. The two steps are: multiplication of the two elements a_{ω_i}, $a_{\omega_{i+1}}$, and reordering of the sequence $(a_0, ..., a_{\sigma(j)}, ..., a_{\sigma(n)})$ into the sequence $(a_{\omega_0}, ..., a_{\omega_j}, ..., a_{\omega_n})$. On both sides we obtain the term

$$(a_{\omega_0}, ..., a_{\omega_i} \circ_i^\sigma a_{\omega_{i+1}}, ..., a_{\omega_n}).$$

This completes the proof. □

3 Comparison between the dialgebra homology HY and its variation HS.

3.1 - Presimplicial set of planar binary trees.[L2] Let Y_n be the set of planar binary trees with $n+1$ leaves and a root. The family $Y = \{Y_n, n \geq 0\}$ form a presimplicial set with face maps $\theta_i : Y_n \longrightarrow Y_{n-1}$, for $i = 0, ..., n$ defined as follows. For any n-tree y, let $\theta_i(y)$ be the $(n-1)$-tree obtained by deleting the i-th leaf from y. Then it is easy to verify that the maps θ_i satisfy the simplicial relations

$$\theta_i\theta_j = \theta_{j-1}\theta_i, \text{ for } i < j.$$

In fact on both sides the maps act on a tree by deleting the i-th and the j-th leaf, in the two possible orders. It is equivalent to delete them simultaneously.

3.2 - Definition. (Dialgebra homology $HHY_*(D)$.) By *dialgebra homology of D (with coefficients in D)* we mean the homology $HHY_*(D)$ of the presimplicial module $CY_n(D) := k[Y_n] \otimes D^{\otimes n+1}$, with face maps $d_i(y \otimes \underline{a}) = \theta_i(y) \otimes d_i^y(\underline{a})$, given, for any $\underline{a} = (a_0, a_1, ..., a_n) \in D^{\otimes n+1}$, by

$$d_i^y(a_0, a_1, ..., a_n) := \begin{cases} (a_0, ..., a_i \circ_i^y a_{i+1}, ..., a_n) & \text{, for } i = 0, ..., n-1, \\ (a_n \circ_n^y a_0, ..., a_{n-1}) & \text{, for } i = n, \end{cases}$$

and products

$$\circ_i^y := \begin{cases} \dashv & \text{if the } i\text{-th leaf is oriented like } \backslash, \\ \vdash & \text{if the } i\text{-th leaf is oriented like } /. \end{cases}$$

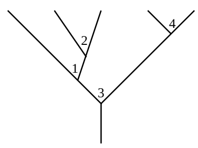
In particular, for any tree y, the operation $\circ_0^y$ is always $\dashv$, and $\circ_n^y$ is always $\vdash$.

(The proof of the simplicial relations for the face maps defined above can be found in [F2].)

The relationship between the homology $HHY_*(D)$ and the $HHS_*(D)$ is given by the trees with levels as follows.

3.3 - Trees with levels. We denote by $\tilde{Y}_n$ the set of planar binary trees with a root and n internal vertices (which is equivalent to having $n+1$ leaves) located, this time, each one of them on a level of height which starts at the root and increases towards the leaves. For example, the tree is not leveled, while and are trees with levels.

Let us chose the following rule to order the internal vertices. Each internal vertex closes a descending path which starts in the *hole* between two consequent leaves and is delimited by edges of the tree. We label an internal vertex by i if its associated hole is located between the leaves number $i-1$ and i. For example:



An n-tree with level can be identified with the permutation which associates to each numbered vertex its level. Thus there is a bijective correspondence between the set S_n of permutations on n elements and the set $\tilde{Y}_n$ of leveled n-trees.

3.4 - Proposition. *There is a natural map on homology*

$$HHS_*(D) \longrightarrow HHY_*(D),$$

which is an isomorphism for $* = 0, 1, 2$.

Proof. In the composite

$$S_n \approx \tilde{Y}_n \longrightarrow Y_n, \quad \text{for } n \geq 0,$$

the second map is a surjection which consists in 'forgetting' the levels of the internal vertices of the tree. The chosen bijection $S_n \approx \tilde{Y}_n$ is such that these maps commute with the simplicial faces on S_n and Y_n. Thus they induce a surjective map of complexes

$$CS_*(D) \longrightarrow CY_*(D).$$

Since for $n = 0, 1, 2$ the sets $\tilde{Y}_n$ and Y_n coincide, and for $n = 3$ the sets $\tilde{Y}_3$ maps surjectively to Y_3, we clearly have an isomorphism on the homology at degree $0, 1$ and 2. $\square$

3.5 - Theorem. *Let A be an associative unital algebra. Then*

$$HHS_*(A) \cong HHY_*(A).$$

Proof. There is a commutative diagram

$$\begin{tikzcd} HHS_*(A) \arrow[rr] \arrow[dr, "\alpha"'] & & HHY_*(A) \arrow[dl, "\beta"] \\ & HH_*(A) & \end{tikzcd}$$

The map $\alpha : HHS_*(A) \longrightarrow HH_*(A)$ is the isomorphism of (1.7), and the map $\beta : HHY_*(A) \longrightarrow HH_*(A)$ is proved to be an isomorphism in [F1]. $\square$

3.6 - Remark. The dialgebra homology HY and its variation HS are not equivalent, in general, since on any abelian dialgebra D (such that $D \dashv D = 0 = D \vdash D$) we obviously have

$$HHY_n(D) = k[Y_n] \otimes D^{\otimes n+1}$$

and

$$HHS_n(D) = k[S_n] \otimes D^{\otimes n+1}$$

for any $n \geq 0$.

References

[C] **Cuvier, C.** *Homologie de Leibniz et Homologie de Hochschild*, Ann. Sc. Ec. Norm. Sup.**27** (1994), 1-45.

[F1] **Frabetti, A.** *Dialgebra homology of associative algebras*, to appear in Note aux C. R. Acad. Sci. Paris.

[F2] **Frabetti, A.** *(Co)omologia delle dialgebre*, (in italian) Ph.D. Thesis, Università di Bologna (Italy) (2/1997).

[F3] **Frabetti, A.** *Homology of disimplicial modules*, In preparation.

[Fr] **Fresse, B.** *Cogroups in Algebras over an Operad are Free Algebras*, Preprint I.R.M.A., (1996).

[H] **Hanlon, P.** *On the complete $GL_n(\mathbf{C})$-decomposition of the stable cohomology of $gl_n(A)$*, Trans. A. M. S. (1) **308** (1988), 209-225.

[K] **Koszul, J.-L.** *Crochet de Schouten-Nijenhuis et Cohomologie*, Soc. Mat. Fr. Asterisque, hors série, (1985), 257-271.

[KS] **Kosmann-Schwarzbach, Y.** *From Poisson Algebras to Gerstenhaber Algebras*, Ann. Inst. Fourier 46-5 (1996) 1241-1272.

[L1] **Loday, J.-L.** *Cyclic Homology*, Grundlehren der math. Wissenschaften **301**, Springer-Verlag, 1992.

[L2] **Loday, J.-L.** *Algèbres ayant deux operations associatives (digèbres)*, C. R. Acad. Sci. Paris **321** (1995), 141-146.

[L-P] **Loday, J.-L. - Pirashvili, T.** *Universal Enveloping Algebras of Leibniz Algebras and (Co-)homology*, Math. Annalen **296** (1993), 569-572.

[L-Pr] **Loday, J.-L. - Procesi, C.** *Homology of symplectic and orthogonal algebras*, Advances in Math. **69** (1988), 93-108.

[L-Q] **Loday, J.-L. - Quillen, D.** *Cyclic Homology and the Lie Algebra Homology of Matrices*, Comment. Math. Helv. **59** (1984), 565-591.

[Lo] **Lodder, J.** *Leibniz Homology and the James Model*, Math. Nachr. **175** (1995), 209-229.

[T] **Tsygan, B. L.** *The homology of matrix Lie algebras over rings and the Hochschild homologie*, (in russian) Uspekhi Mat. Nauk **38** (1983), 217-218; (in english) Russ. Math. Survey **38** No. 2 (1983), 198-199.

Dipartimento di Matematica
dell'Università di Bologna,
Piazza di Porta San Donato, 5
40127 Bologna, Italy
e-mail : frabettidm.unibo.it

Partially supported by the European project "K-theory" $SCI^* - CT91 - 0756$.