Dialgebra (co)homology with coefficients

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Abstract

Dialgebras are a generalization of associative algebras which gives rise to Leibniz algebras instead of Lie algebras. In this paper we define the dialgebra (co)homology with coefficients, recovering, for constant coefficients, the natural bar homology of dialgebras introduced by J.-L. Loday in [L6] and denoted by HY_*. We show that the homology HY_* has the main expected properties: it is a derived functor, HY^2 classifies the abelian extensions of dialgebras and Morita invariance of matrices holds for bar-unital dialgebras (the best analogue to unital associative algebras). For associative algebras, we compare Hochschild and dialgebra homology, and extend the isomorphism proved in [F2] for unital algebras to the case of H-unital algebras.

A feature of the theory HY is that the categories of coefficients for homology and cohomology are different. This leads us to introduce the universal enveloping algebra of dialgebras and the corresponding cotangent complex, analogue to that defined by D. Quillen for commutative algebras. Our results follow from a property of Poincaré-Birkhoff-Witt type and from some combinatorial and simplicial properties of the sets of planar binary trees proved in [F4]. Finally, remarking that for bar-unital dialgebras the faces and degeneracies satisfy all the simplicial relations except one, leads us to study the general properties of the so-called *almost simplicial modules*.

Introduction

It is well known that any associative algebra gives rise to a Lie algebra, with bracket $[a,b] := ab - ba$. In [L2], J.-L. Loday introduces a non-antisymmetric version of Lie algebras, whose bracket satisfies the Leibniz relation

$$(L) \quad [x,[y,z]] = [[x,y],z] - [[x,z],y],$$

and therefore called *Leibniz algebras*. The Leibniz relation, combined with antisymmetry, is a variation of the Jacobi identity, hence Lie algebras are antisymmetric Leibniz algebras. The natural homology theory of Leibniz algebras is denoted by HL_*.

In [L5], Loday introduces an 'associative' version of Leibniz algebras, called *dialgebras*, equipped with two binary operations, $\dashv$ and $\vdash$, which satisfy the five relations

$$(D) \quad \begin{cases} a \dashv (b \dashv c) = (a \dashv b) \dashv c = a \dashv (b \vdash c), \\ (a \vdash b) \dashv c = a \vdash (b \dashv c), \\ (a \dashv b) \vdash c = a \vdash (b \vdash c) = (a \vdash b) \vdash c. \end{cases}$$

These identities are all variations of the associative law, so associative algebras are dialgebras for which the two products coincide. The peculiar point is that the bracket $[a,b] := a \dashv b - b \vdash a$

defines a Leibniz algebra which is not antisymmetric, unless the left and right products coincide. Hence dialgebras yield a commutative diagram of categories and functors

$$\begin{tikzcd} \mathbf{Alg}_k \arrow[r, "-"] \arrow[d] & \mathbf{Lie}_k \arrow[d] \\ \mathbf{Dig}_k \arrow[r, "-"] & \mathbf{Leib}_k \end{tikzcd} .$$

A peculiar example of a dialgebra is the *derived dialgebra* of an associative differential algebra (A, d), with products

$$a \dashv b := a \cdot d(b) \qquad \text{and} \qquad a \vdash b := d(a) \cdot b, \qquad a, b \in A.$$

Its associated Leibniz algebra coincides with the derived Leibniz algebra of the differential Lie algebra $(A_{Lie}, [\ ,\ ], d)$, with bracket $[x, y]_d := [x, dy]$. In [KS], Y. Kosmann-Schwarzbach exhibits some examples of derived Leibniz algebras from classical differential geometry and physics.

The natural bar homology theory for dialgebras, found by J.-L. Loday in [L5], is denoted by HY_*, because the module of n-chains of a dialgebra D is given by $k[Y_n] \otimes D^{\otimes n}$, where Y_n is the set of planar binary trees with n internal vertices.

In this paper we define the dialgebra homology and cohomology *with coefficients* and perform some computations. For constant coefficients we recover the bar homology defined by Loday.

In order to determine the coefficients, we consider an extension of a dialgebra D by an abelian dialgebra M. Then M inherits some actions of D which make it into a *representation* of D. Moreover, there is a natural choise of the first terms of a cochain complex $CY^1(D, M) \xrightarrow{\delta} CY^2(D, M) \xrightarrow{\delta} CY^3(D, M)$ such that the second cohomology group classifies the abelian extensions of the dialgebra D by M,

$$HY^2(D, M) \cong \mathrm{Ext}(D, M).$$

We extend the definition of the coboundary operator to higher terms, obtaining the dialgebra cohomolgy groups with coefficients, $HY^n(D, M)$, for any $n \geq 0$.

In order to determine the coefficients for the homology, we interpretate the representations of D as left modules over a unital associative algebra $E(D)$ called *universal enveloping algebra*. (For an associative algebra A, the analogue algebra is $A^e = A \otimes A^{op}$.) Then the coefficients for the homology of D are taken to be the right $E(D)$-modules and they are called *corepresentations* of D. The cochain complex of D with any coefficients can be dualized into a chain complex with coefficients in the universal enveloping algebra $E(D)$. In analogy with a complex defined by D. Quillen for commutative algebras, we call it *cotangent complex* of D. The chain complex of D with coefficients in a corepresentation N is then obtained by considering the tensor product over $E(D)$ of N and the cotangent chain complex. The resulting homology groups are denoted by $HY_n(D, N)$.

We prove that dialgebra homology and cohomology are derived functors,

$$HY_*(D, N) \cong \mathrm{Tor}_*^{E(D)}(N, M(D)), \qquad HY^*(D, M) \cong \mathrm{Ext}^*_{E(D)}(M(D), M).$$

In fact, using a property of Poincaré-Birkhoff-Witt type on the universal enveloping algebra $E(D)$ and some properties of the set of planar binary trees proved in [F4], we prove that the cotangent complex is acyclic, with $HY_0(D, E(D)) \cong M(D)$.

Comparing Hochschild and dialgebra homology of associative algebras, using similar arguments, we generalize the isomorphism proved in [F2] for unital associative algebras to the case of H-unital algebras,

$$HY_*(A, M) \cong H_*(A, M).$$

We also introduce the notion of HY-unital dialgebras and show that for associative algebras this property is equivalent to the H-unitality in the sense of Wodzicki, [W].

Both Hochschild and dialgebra chain complexes have boundary operator given by means of some face maps. For associative algebras, a unit permits us to define some degeneracy maps, and the Hochschild chain complex is in fact a simplicial module. The best analogue of a unit, for dialgebras, is an element which is a unit only on the bar-side of the products (bar-unit), since dialgebras with unit are in fact associative algebras. The bar-unit enables us to define some degeneracy maps which satisfy all the simplicial relations except one, namely $s_i s_i \neq s_{i+1} s_i$, and therefore they are called *almost-simplicial*. Using some simplicial techniques described in [F2], we show that bar-unital dialgebras have homological dimension ≤ 0, that is,

$$HY_n(D, N) = 0, \qquad n \geq 1,$$

that they allow Morita invariance of matrices for dialgebra homology,

$$HY_*(\mathrm{M}_r(D), \mathrm{M}_r(N)) \cong HY_*(D, N),$$

and also a weak version of the Künneth formula for dialgebra homology.

The paper is organized as follows. In section 1 we review the basic definitions of dialgebras and give a characterization of any dialgebra as a bimodule over its canonical associative algebra. In section 2 we define the dialgebra cohomology and prove that the second group classifies the abelian extensions of dialgebras. In section 3 we define the universal enveloping algebra and prove the Poincaré-Birkhoff-Witt isomorphism on its associated graded algebra. In section 4 we define the cotangent chain complex and the dialgebra homology. In section 5 we use some properties of the set of planar binary trees to define a spectral sequence abutting to the dialgebra homology. We then prove the Tor interpretation. With the same techniques, in section 7 we show that for associative algebras the notions of HY-unitality and that of H-unitality are equivalent. In section 6 we study the homology of bar-unital dialgebras, using simplicial tools.

Contents

1 Dialgebras

In this section we review the main definition and give a characterization of any dialgebra as a bimodule over its canonical associative algebra. More details and examples can be found in [L5].

1.1 - Dialgebras. Let k be a field. A *dialgebra over k* is a k-vector space D with two operations, $\dashv, \vdash : D \otimes D \longrightarrow D$, called left and right products, satisfying the following five axioms:

$$(D) \quad \begin{cases} a \dashv (b \dashv c) \overset{1}{=} (a \dashv b) \dashv c \overset{2}{=} a \dashv (b \vdash c), \\ (a \vdash b) \dashv c \overset{3}{=} a \vdash (b \dashv c), \\ (a \dashv b) \vdash c \overset{4}{=} a \vdash (b \vdash c) \overset{5}{=} (a \vdash b) \vdash c, \end{cases} \qquad a, b, c \in D.$$

A dialgebra is called *bar-unital* if it is given a specified *bar-unit*: an element $1 \in D$(not necessarily unique) which is a unit for the left and right products only on the bar-side, that is $1 \vdash a = a = a \dashv 1$, for any $a \in D$. A *morphism of dialgebras* is a k-linear map $f : D \longrightarrow D'$ which preserves the products, i.e. such that $f(a \star b) = f(a) \star f(b)$, where $\star$ denotes either the product $\dashv$ or the product $\vdash$. It is called *unital* if the image of any bar-unit is a bar-unit. Denote by $\mathbf{Dig}_k$ the category of dialgebras over k, and by $\mathbf{uDig}_k$ that of bar-unital dialgebras.

Any associative (unital) algebra A is clearly a (bar-unital) dialgebra, that we denote by A_{di}, for which the left and right products coincide:

$$a \dashv b := a \cdot b =: a \vdash b, \qquad a, b \in A.$$

The category $\mathbf{Alg}_k$ of associative algebras over k is a full subcategory of $\mathbf{Dig}_k$, and similarly that of unital associative algebras $\mathbf{uAlg}_k$ is a full subcategory of $\mathbf{uDig}_k$.

1.2 - Canonical Leibniz algebra of a dialgebra. Recall, from [L2], that a *Leibniz algebra* is a k-module $\mathfrak{g}$ equipped with a bracket $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ satisfying the Leibniz relation

$$(L) \quad [x, [y, z]] = [[x, y], z] - [[x, z], y], \qquad x, y, z \in \mathfrak{g}.$$

Then Lie algebras are antisymmetric Leibniz algebras. In [L5], Loday observed that, given a dialgebra D, the bracket

$$[a, b] := a \dashv b - b \vdash a, \qquad a, b \in D,$$

defines a Leibniz algebra D_L which is not antisymmetric, unless the left and right products coincide. The assignement $D \mapsto D_L$ yields a commutative diagram of categories and functors

$$\begin{tikzcd} \mathbf{Alg}_k \arrow[r, "-"] \arrow[d] & \mathbf{Lie}_k \arrow[d] \\ \mathbf{Dig}_k \arrow[r, "-"] & \mathbf{Leib}_k \end{tikzcd} .$$

1.3 - Example. An interesting example of a dialgebra, which is not an associative algebra, is constructed on an associative differential algebra (A, d), by introducing the *derived products*

$$a \dashv b := a \cdot d(b) \qquad \text{and} \qquad a \vdash b := d(a) \cdot b, \qquad a, b \in A,$$

(with an appropriate sign in the case of graded algebras). The assumptions on d imply the relation $d(a\ db) = da\ db = d(da\ b)$, from which follow immediately the dialgebra axioms (D) for $\dashv$ and $\vdash$. We call *derived dialgebra* of (A, d) the triple $A_d := (A, \dashv, \vdash)$.

If the algebra A is endowed with a unit 1, and the field k has characteristic different from 2, then necessarily $d1 = 0$. Hence, in the derived dialgebra A_d, we have $a \dashv 1 = a \cdot d1 = 0$ and similarly $1 \vdash a = d1 \cdot a = 0$. This means that the element 1 is not a bar-unit in the dialgebra A_d. A bar-unit, in A_d, is an element $x \in A$ such that $dx = 1$, and it does not necessarily exist.

The canonical Leibniz algebra of the derived dialgebra A_{di} is an example of a Leibniz algebra, which is not Lie, whose role in classical differential geometry has recently been pointed out by Y. Kosmann-Schwarzbach. In [KS], she observes that a differential Lie algebra $(\mathfrak{g}, [\ ,\ ], d)$, always gives rise to the *derived Leibniz algebra* $\mathfrak{g}_d$, by introducing the *derived bracket*

$$[x, y]_d := [x, dy], \qquad x, y \in \mathfrak{g},$$

which satisfies (L) but it is not antisymmetric. In fact, if $(\mathfrak{g}, d)$ is the differential Lie algebra of (A, d), with $\mathfrak{g} = A_L$, its derived Leibniz algebra $\mathfrak{g}_d$ coincides with the Leibniz algebra of A_d, i.e. $(A_d)_L \cong (A_L)_d$.

1.4 - Canonical associative algebra of a dialgebra. Let D be a dialgebra and set

$$D_{as} := D/Z(D), \qquad Z(D) := \langle a \dashv b - a \vdash b \mid a, b \in D \rangle,$$

where $Z(D)$ is an ideal in D, i.e. a k-submodule closed under multiplication for any element of D. Then D_{as} is an associative algebra, with product $[a] \cdot [b] := [a \dashv b] = [a \vdash b]$. For example, in the free dialgebra $D = \bar{D}ias(V)$ on a vector space V, as defined in [L5], the fusion maps any element $\underline{v} \dashv \underline{w} - \underline{v} \vdash \underline{w} \in D$ to $\underline{vw} - \underline{vw} = 0$. Therefore it induces an isomorphism with the free associative algebra $\bar{D}ias(V)_{as} \cong \bar{A}ss(V)$.

If A is an associative algebra, since $Z(A_{di}) = 0$, there is a natural isomorphism of algebras $(A_{di})_{as} \cong A$. Hence the functor di of (1.1) admits a left adjoint functor $as : \mathbf{Dig}_k \longrightarrow \mathbf{Alg}_k$.

When D has a bar-unit 1, the class $[1]$ is a unit in D_{as}, hence $as : \mathbf{uDig}_k \longrightarrow \mathbf{uAlg}_k$. In this case, the ideal $Z(D)$ is generated by the elements $a - a \vdash 1$, $a - 1 \dashv a$ for any $a \in D$. If, in addition, the bar-unit is also a unit on the pointed side of one product, this ideal is zero and $D = D_{as}$ is a unital associative algebra.

1.5 - Dialgebra induced by a bimodule morphism. In [L5] Loday exhibits a typical example of a dialgebra, constructed on a bimodule M over an associative algebra A. If there is a morphism of A-bimodules $f : M \longrightarrow A$, then on M there are dialgebra products

$$m \dashv m' := m \cdot f(m'), \qquad m \vdash m' := f(m) \cdot m', \qquad m, m' \in M.$$

We now invert this procedure and show that all dialgebra can be achieved in this way.

1.6 - Proposition. *Let D be a dialgebra, and D_{as} its canonical associative algebra. Then there exists a D_{as}-bimodule structure on D, and a morphism of D_{as} bimodules $f : D \longrightarrow D_{as}$ such that the dialgebra structure on D is recovered by $a \dashv b = a \cdot f(b)$ and $a \vdash b = f(a) \cdot b$.*

Proof. First define on D a structure of bimodule over D_{as}, by setting

$$D_{as} \otimes D \longrightarrow D, \quad [a] \cdot b := a \vdash b \qquad \text{and} \qquad D \otimes D_{as} \longrightarrow D, \quad a \cdot [b] := a \dashv b.$$

The identities 1, 2 and 4, 5 of (D) guarantee that the equality $[a \dashv b] = [a \vdash b]$ in D_{as} is coherent with the above actions. In respect to these actions, the canonical projection $D \longrightarrow D_{as}$ is clearly a bimodule morphism and it induces on D precisely the given dialgebra structure. **2**

Before closing this section we give a definition which is frequently used in the sequel.

1.7 - Abelian dialgebras and abelianization. A dialgebra D is called *abelian* if it has zero products, i.e. $a \dashv b = 0 = a \vdash b$ for any $a, b \in D$. Any k-module can be seen as an abelian dialgebra, by setting all products to zero. Another example is the ideal $Z(D)$, which is an abelian sub-dialgebra of D. In fact, the identities 1,2 of (D) show that $D \dashv Z(D) = 0$, so in particular $Z(D) \dashv Z(D) = 0$. Similarly, the identities 4,5 show that $Z(D) \vdash Z(D) = 0$.

Given a dialgebra D, we call *abelianization of* D the abelian dialgebra

$$D_{ab} := D/D^2, \qquad D^2 := \langle a \dashv b, a \vdash b \mid a, b \in D \rangle,$$

where D^2 is the ideal generated by all products. For instance, in a free dialgebra $\bar{D}ias(V) = \bar{T}(V) \otimes V \otimes \bar{T}(V)$ the ideal $\bar{D}ias(V)^2$ consists of the whole dialgebra except the central elements, V. Hence there is an isomorphism of abelian dialgebras $\bar{D}ias(V)_{ab} \cong V$.

When D has a bar-unit 1, the ideal D^2 is generated by the elements a, $a \vdash 1$ and $1 \dashv a$ for any $a \in D$. Therefore $D^2 := \langle a, a \vdash 1, 1 \dashv a \mid a \in D \rangle \cong D$ and $D_{ab} = 0$.

2 Dialgebra cohomology and representations

In this section we define the dialgebra cohomology with coefficients in a representation, in such a way that the second cohomology group classifies the abelian extensions of dialgebras.

The boundary operator of the bar chain complex, in low dimension, only depends on the axioms of dialgebra. This means that, given a dialgebra D, the composition $d'^2 : D^{\otimes 3} \longrightarrow D^{\otimes 2} \longrightarrow D$ yields zero because of relations (D). Hence the module of 3-chains is made of 5 copies of $D^{\otimes 3}$, one for each identity of (D), and the module of 2-chains is made of 2 copies of $D^{\otimes 2}$, one for each operation. In [L5], Loday extends this complex by considering, in dimension n, a number of copies of $D^{\otimes n}$ equal to the Catalan number $c_n = \frac{2n!}{n!(n+1)!}$. The boundary operator d' can be easily described in terms of simplicial faces d_i, that is, $d' = \sum_{i=1}^{n-1} (-1)^i d_i$, if we regard the Catalan number as the cardinality of the set Y_n of planar binary trees with n internal vertices. Hence the dialgebra homology is denoted by HY_*.

2.1 - Representations. Let D be a dialgebra over a field k. A *representation* of D is a k-module M endowed with two right actions $\dashv, \vdash$: $M \otimes D \longrightarrow M$ and two left actions $\dashv, \vdash$: $D \otimes M \longrightarrow M$ satisfying the 15 axioms given by the 5 axioms (D) for each choice of one of the three elements a, b or c in M. We refer to these relations as (RD). If D has a bar-unit 1, a representation M is called *unital* if the identity $1 \vdash m = m = m \dashv 1$ holds for any $m \in M$.

If D_L is the canonical Leibniz algebra of D, on the k-module M we obtain a representation M_L of the Leibniz algebra D_L, as defined in [L-P], by introducing the new brackets

$$[m, a] := m \dashv a - a \vdash m \qquad \text{and} \qquad [a, m] := a \dashv m - m \vdash a, \qquad m \in M, \quad a \in D.$$

2.2 - Examples.

1. Any dialgebra D is clearly a representation of itself, and it is a unital representation if D has a bar-unit. Similarly, any ideal I is a representation of D.

2. Any k-module M becomes a representation of a dialgebra D by setting $m \star a := 0 =: a \star m$, for any $m \in M$ and $a \in D$. We call M the *trivial* representation of D. When D is bar-unital, the trivial representation can not be unital, unless $M = 0$.

3. Let D be a dialgebra and $D_{as} = D/Z(D)$ its canonical associative algebra. Then the ideal $Z(D)$ is a representation of the dialgebra $(D_{as})_{di}$,

   with actions

$$z \star [a] := z \star a \qquad \text{and} \qquad [a] \star z := a \star z, \qquad z \in Z(D), \quad [a] \in (D_{as})_{di}.$$

   In fact the choise of the representative element $a \in D$ for the class $[a] \in (D_{as})_{di}$ is not relevant, because any other representant a' differs from a for an element z' of $Z(D)$, $a = a' + z'$, and since $Z(D)$ is an abelian dialgebra we have $z \star a = z \star a' + z \star z' = z \star a'$.

4. Given a dialgebra D with a representation M, the k-module $\mathrm{M}_r(M)$ of matrices with entries in M is a representation of the dialgebra $\mathrm{M}_r(D)$, with 'row $\times$ column' actions.

2.3 - Dialgebra cochain complex. Let D be a dialgebra over k and M a representation of D. For any $n \geq 0$, we call *module of n-cochains of D with coefficients in M* the k-module

$$CY^n(D,M) := \mathrm{Hom}_k(k[Y_n] \otimes D^{\otimes n}, M),$$

where Y_n is the set of planar binary trees with n internal vertices, see /citeL:dig. In particular, since the sets Y_0 and Y_1 contain one tree, respectively $|$ and $\vee$, the module of 0-cochains is $CY^0(D,M) = \mathrm{Hom}_k(k,M) \cong M$ and that of 1-cochains is $CY^1(D,M) = \mathrm{Hom}_k(D,M)$.

For any $n \geq 0$, we define the *coboundary operator* $\delta : CY^n(D,M) \longrightarrow CY^{n+1}(D,M)$ as the k-linear map $\delta = \sum_{i=0}^{n+1} (-1)^i \delta^i$ given by means of the following face maps:

$$(\delta^i f)(y \otimes (a_1, ..., a_{n+1})) := \begin{cases} a_1 \circ_0^y f(d_0(y) \otimes (a_2, ..., a_n)), & \text{for } i = 0, \\ f(d_i(y) \otimes (a_1, ..., a_i \circ_i^y a_{i+1}, ..., a_{n+1})), & \text{for } i = 1, ..., n, \\ f(d_{n+1}(y) \otimes (a_1, ..., a_n)) \circ_{n+1}^y a_{n+1}, & \text{for } i = n+1, \end{cases}$$

for any $y \in Y_{n+1}$, $a_1, ..., a_{n+1} \in D$ and $f : k[Y_n] \otimes D^{\otimes n} \longrightarrow M$, where the n-tree $d_i(y)$ is obtained by *deleting* the i^{th} leaf from y, and the maps $\circ_i : Y_{n+1} \longrightarrow \{\dashv, \vdash\}$ are defined by

$$\circ_0^y := \begin{cases} \dashv, & \text{if } y \text{ has the shape }, \\ \vdash, & \text{if } y \text{ has the shape }, \end{cases}$$
$$\circ_i^y := \begin{cases} \dashv, & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like } \backslash, \\ \vdash, & \text{if the } i^{th} \text{ leaf of } y \text{ is oriented like } /, \end{cases} \qquad i = 1, ..., n$$
$$\circ_{n+1}^y := \begin{cases} \vdash, & \text{if } y \text{ has the shape }, \\ \dashv, & \text{if } y \text{ has the shape }. \end{cases}$$

2.4 - Lemma. *The operators $\delta : CY^n(D,M) \longrightarrow CY^{n+1}(D,M)$ satisfy the identity $\delta \circ \delta = 0$.*

Proof. It suffices to verify that the face maps δ^i satisfy the cosimplicial relations $\delta^j \delta^i = \delta^i \delta^{j-1}$, for $i < j$. In particular, since for $i = 1, ..., n-1$ the faces $\delta^i : CY^n(D,M) \longrightarrow CY^{n+1}(D,M)$ are the duals of the faces $d_i : CY_{n+1}(D) \longrightarrow CY_n(D)$, defined in [L5], which satisfy the simplicial relations $d_i d_j = d_{j-1} d_i$, it suffices to show that these relations hold for $i = 0$ with j from 1 to $n+2$, and for $j = n+2$ with i from 0 to $n+1$.

Let us verify that $\delta^j \delta^0 = \delta^0 \delta^{j-1}$ for $j = 1, ..., n+2$. Take a k-linear map $f : k[Y_n] \otimes D^{\otimes n} \longrightarrow M$ and an element $y \otimes \underline{a} = y \otimes (a_1, ..., a_{n+2}) \in k[Y_{n+2}] \otimes D^{\otimes n+2}$. Then for $j = 1$ we have on one side:

$$\delta^1 \delta^0 f(y \otimes \underline{a}) = (a_1 \circ_1^y a_2) \circ_0^{d_1(y)} f(d_0 d_1(y) \otimes \underline{a}') = \begin{cases} (a_1 \dashv a_2) \dashv m, & \text{if } y = \\ (a_1 \vdash a_2) \dashv m, & \text{if } y = \\ (a_1 \star a_2) \vdash m, & \text{otherwise} \end{cases}$$

where $m := f(d_0 d_1 \otimes \underline{a'})$. Notice that the symbol $\star$ here means that for some trees we have the product $\dashv$, for some others we have $\vdash$, but because of equalities 4 and 5 in (D) the final product always agrees in the term $(a_1 \vdash a_2) \vdash m$. On the other side:

$$\delta^0 \delta^0 f(y \otimes \underline{a}) = a_1 \circ_0^y [a_2 \circ_0^{d_0(y)} f(d_0 d_0(y) \otimes \underline{a'})] = \begin{cases} a_1 \dashv (a_2 \star' m), & \text{if } y = \Yleft \\ a_1 \vdash (a_2 \dashv m), & \text{if } y = \Yright \\ a_1 \vdash (a_2 \vdash m), & \text{otherwise} \end{cases}$$

where we have identified $f(d_0 d_0(y) \otimes \underline{a'})$ with $f(d_0 d_1(y) \otimes \underline{a'}) = m$ because of the identity $d_0 d_1 = d_0 d_0$. Hence the equality of the two terms is a consequence of the axioms of the representation M. For $j > 1$, we have on one side:

$$\delta^j \delta^0 f(y \otimes \underline{a}) = a_1 \circ_0^{d_j(y)} f(d_0 d_j(y) \otimes (a_2, ..., a_j \circ_j^y a_{j+1}, ..., a_{n+2}),$$

and on the other side:

$$\delta^0 \delta^{j+1} f(y \otimes \underline{a}) = a_1 \circ_0^y f(d_{j-1} d_0(y) \otimes (a_2, ..., a_j \circ_{j-1}^{d_0(y)} a_{j+1}, ..., a_{n+2}).$$

Hence the equality of the two terms is a consequence of the identities $d_0 d_j = d_{j+1} d_0$, $\circ_0^{d_j(y)} = \circ_0^y$ and $\circ_{j-1}^{d_0(y)} = \circ_j^y$ for any $j > 1$.

In a similar way one verifies that $\delta^{n+2} \delta^i = \delta^i \delta^{n+1}$ for $i = 0, ..., n+1$. **2**

2.5 - Definition (dialgebra cohomology). Let D be a dialgebra with a representation M. The *dialgebra cohomology of D with coefficients in M* is the homology of the cochain complex (2.3),

$$HY^n(D, M) := H_n(CY^*(D, M), \delta), \qquad n \geq 0.$$

In particular, when $M = D$, we use the notation $HHY^*(D) := HY^*(D, D)$.

2.6 - Derivations on dialgebras. Let D be a dialgebra over a field k, and M a representation of D. A *derivation on D with values in M* is a k-linear map $\mu : D \longrightarrow M$ such that

$$\mu(a \star b) = \mu(a) \star b + a \star \mu(b), \qquad a, b \in D.$$

Denote by $\mathrm{Der}(D, M)$, the k-module of all derivations on D with values in M. The *inner derivations of D in M* are the adjoint maps $ad_m = [-, m] : D \longrightarrow M$, for $m \in M$, and their set is $\mathrm{Inn}(D, M) = \{ad_m \in \mathrm{Der}(D, M) |$ for all $m \in M\}$. (Notice that, unlike for associative algebras, the map $[m, -] : D \longrightarrow M$ is *not* a derivation of D!)

2.7 - Elementary computations. Given a dialgebra D with a representation M, the zero-th coboundary of D in M is the k-linear map $\delta : M \longrightarrow \mathrm{Hom}(D, M)$, $\delta m(a) := a \dashv m - m \vdash a = [a, m]$. Hence $HY^0(D, M)$ is the submodule of invariants of M,

$$HY^0(D, M) = M^D = \{m \in M | \ [a, m] = 0 \text{ for any } a \in D\}.$$

The first coboundary is the k-linear map $\delta : Hom_k(D, M) \longrightarrow Hom_k(k[Y_2] \otimes D^{\otimes 2}, M)$,

$$(\delta g)(\Yleft \otimes (a, b)) := g(a) \dashv b - g(a \dashv b) + a \dashv g(b),$$
$$(\delta g)(\Yright \otimes (a, b)) := g(a) \vdash b - g(a \vdash b) + a \vdash g(b).$$

Thus the 1-cocycles of D with values in M are the derivations, while the 1-coboundaries are the inner derivations. Hence we have the classical result

$$HY^1(D,M) \cong \mathrm{Der}(D,M)/\mathrm{Inn}(D,M).$$

Notice that we established a bijection between the set Y_2 of 2-trees and the set $\{\dashv, \vdash\}$ of binary operations of a dialgebra: the trees , correspond respectively to the products $\dashv$, $\vdash$.

2.8 - Abelian extensions of dialgebras. Let D be a dialgebra over k. By definition, an *abelian extension* of D is an exact sequence of dialgebras

$$0 \longrightarrow M \stackrel{i}{\longrightarrow} D' \stackrel{\pi}{\longrightarrow} D \longrightarrow 0,$$

which is split as a sequence of k-modules and where $M = \ker\pi$ is an abelian dialgebra. The first assumption forces the dialgebra D' to have underlying k-module $M \oplus D$, with inclusion $i(m) = (m,0)$, projection $\pi(m,a) = a$, sections $\sigma(a) = (g(a),a)$ for arbitrary k-linear maps $g : D \longrightarrow M$, and products of generic form

$$(m,a) \star (n,b) := (m \star b + a \star n + f(\star; a,b), a \star b), \qquad m,n \in M, \quad a,b \in D,$$

where the k-linear map $f\colon k[\dashv,\vdash] \otimes D^{\otimes 2} \longrightarrow M$ must satisfy a 'cocycle condition' given in (2.9). The second assumption naturally endows M with the structure of a D-representation,

$$m \star a := i(m) \star a' \qquad \text{and} \qquad a \star m := a' \star i(m), \qquad m \in M, a \in D,$$

where $a' \in D'$ is such that $\pi(a') = a$. Hence the element a' must be of the form (m', a), for some $m' \in M$, where it is clear that any choise of $m' \neq 0$ does not affect the above definition.

Given a representation M of D, an *abelian extension of D by M* is an abelian extension such that the structure of D-representation induced on M coincides with the given one. Two abelian extensions of D by M, say D' and D'', are *equivalent* if there exists a dialgebra map $D' \longrightarrow D''$ which is compatible with the identity on M and on D. We denote by $\mathrm{Ext}(D,M)$ the set of equivalence classes of extensions of D by M.

An abelian extension of D by M is called *trivial* if it is split as a sequence of dialgebras. The prototype is the dialgebra $M \oplus D$ with products associated to the zero map, $f = 0$:

$$(m,a) \star (n,b) := (m \star b + a \star n, a \star b).$$

In fact the short exact sequence $0 \longrightarrow M \longrightarrow M \oplus D \longrightarrow D \longrightarrow 0$ is surely split, with section $\sigma : D \longrightarrow M \oplus D$ defined by $\sigma(a) := (0,a)$.

2.9 - Theorem. *Let D be a dialgebra over k and M a representation of D. Then the map which associates a 2-cocycle f to the abelian extension of D by M given by the dialgebra $M \oplus D$, with products defined by f, yields a natural isomorphism*

$$HY^2(D,M) \cong \mathrm{Ext}(D,M).$$

Proof. 1. Let $f : k[Y_2] \otimes D^{\otimes 2} \longrightarrow M$ be a k-linear map. Using the bijection $\{\dashv,\vdash\} \simeq Y_2$ described above, consider the products on $M \oplus D$ defined by f. Fix the following bijection: the axioms 1, 2, 3, 4, 5 of (D) correspond respectively to the trees , , , , of Y_3. Then the five axioms (D) for the dialgebra $M \oplus D$ produce five conditions on the map f which are equivalent to the condition $\delta(f) = 0$. Hence the products defined on $M \oplus D$ by f determine a structure of abelian extension of D by M if and only if f is a 2-cocycle.

2. The abelian extension $M \oplus D$, with products defined by f, is trivial if and only if there exists a section $\sigma : D \longrightarrow M \oplus D$ which is a morphism of dialgebras, i.e. $\sigma(a \star b) = \sigma(a) \star \sigma(b)$ for any $a, b \in D$. The section must be of the form $\sigma(a) = (g(a), a)$, for a k-linear map $g : D \longrightarrow M$. Then, by definition of the products on $M \oplus D$ associated to f, the map g must satisfy the condition

$$g(a \star b) = g(a) \star b + a \star g(b) + f(\star; a, b), \qquad a, b \in D.$$

This is essentially $\delta(g) = f$. Hence the extension is trivial if and only if f is a 2-coboundary.

3. It remains to show that homology classes of 2-cocycles correspond bijectively to equivalence classes of abelian extensions. In other words, that when a 2-cocycle f is modified with a coboundary $\delta(g)$ into a new 2-cocycle $f' = f + \delta(g)$, the two corresponding extensions are equivalent. This is immediately proved, with equivalence given by the map $M \oplus D \longrightarrow (M \oplus D)'$: $(m, a) \mapsto (m + g(a), a)$. The inverse follows from step 2. **2**

2.10 - Characteristic element of a dialgebra. Let D be a dialgebra and $D_{as} = D/Z(D)$ its canonical associative algebra. The ideal $Z(D)$ is an abelian sub-dialgebra of D and a representation over $(D_{as})_{di}$. Hence the short exact sequence

$$0 \longrightarrow Z(D) \longrightarrow D \longrightarrow (D_{as})_{di} \longrightarrow 0$$

presents the dialgebra D as abelian extension of $(D_{as})_{di}$ by $Z(D)$. Its class in $HY^2((D_{as})_{di}, Z(D))$ is called *characteristic element* of the dialgebra D, and denoted by $ch(D)$.

3 Universal enveloping algebra of dialgebras

The duality between homology and cohomology groups is reflected on their modules of coefficients. For associative and Lie algebras these are both bimodules. For Leibniz algebras, the category of representations (coefficients for cohomology) is not equivalent to its dual category: the dual modules are called corepresentations (see [L-P]). The same happens for dialgebras.

To define the corepresentations, for a dialgebra D, we introduce a unital associative algebra $E(D)$ such that the category of representations of D is isomorphic to the category of left modules over $E(D)$. Then the corepresentations of D are the right $E(D)$-modules.

The universal enveloping algebra of dialgebras has a property of Poincaré-Birchoff-Witt type (see (3.8)), which plays a fundamental role in the computation of dialgebra homology as a derived functor (theorem (5.1)).

3.1 - Definition (universal enveloping algebra). Let D be a dialgebra. By definition, the *universal enveloping algebra* $E(D)$ is the quotient of the free unital associative algebra $T(Dx \oplus Du \oplus Dy \oplus Dv)$ on four copies of D, labeled by the symbols x, u, y, v, by the ideal corresponding to the following 15 relations

$$(ED)\quad \begin{cases} x(a)\cdot x(b) \overset{1}{=} x(a\dashv b) \overset{2}{=} x(a\vdash b), \\ y(a)\cdot y(b) \overset{3}{=} y(b\dashv a) \overset{4}{=} y(b\vdash a), \\ x(a)\cdot y(b) \overset{5}{=} y(b)\cdot x(a), \\ x(a)\cdot u(b) \overset{6}{=} u(a\vdash b), \\ u(a)\cdot u(b) \overset{7}{=} u(a)\cdot x(b) \overset{8}{=} u(a\dashv b), \\ y(a)\cdot v(b) \overset{9}{=} v(b\dashv a), \\ v(a)\cdot v(b) \overset{10}{=} v(a)\cdot y(b) \overset{11}{=} v(b\vdash a), \\ x(a)\cdot v(b) \overset{12}{=} v(b)\cdot x(a) \overset{13}{=} v(b)\cdot u(a), \\ y(a)\cdot u(b) \overset{14}{=} u(b)\cdot y(a) \overset{15}{=} u(b)\cdot v(a), \end{cases} \qquad a,b\in D.$$

If the dialgebra D has a bar-unit 1, the element $x(1)=y(1)=1_E$ is the unit of $E(D)$, while $u(1)$ is only a right unit for $u(a)$ and $v(1)$ is only a right unit for $v(a)$, for any $a\in D$.

3.2 - Proposition. *A k-module M is a (unital) representation of a (bar-unital) dialgebra D if and only if it is a (unital) left module over $\bar{E}(D)$.*

Proof. Given a D-representation M, one defines a left action of $E(D)$ on M, and conversely, by setting

$$\begin{matrix} x(a)\cdot m := a\vdash m, & y(a)\cdot m := m\dashv a, \\ u(a)\cdot m := a\dashv m, & v(a)\cdot m := m\vdash a, \end{matrix} \qquad m\in M, \quad a\in D.$$

The same formulas hold for unital D-representations and unital left $E(D)$-modules. **2**

3.3 - Symmetric and antisymmetric enveloping algebras. Let D be a dialgebra. Call *symmetric enveloping algebra* the quotient

$$S(D) := E(D)/I(D), \qquad I(D) := \langle x(a)-u(a), y(a)-v(a) \mid a\in D\rangle.$$

A representation M of D is called *symmetric* if it is a left module over $S(D)$, i.e. if the left action $E(D)\otimes M \longrightarrow M$ passes to the quotient $E(D)\longrightarrow S(D)$ ($\Longleftrightarrow I(D)\cdot M = 0$). This is equivalent to require that M satisfies the conditions

$$(SRD)\quad \begin{cases} m\dashv a - m\vdash a = 0, \\ a\dashv m - a\vdash m = 0, \end{cases} \qquad m\in M, \quad a\in D.$$

Conversely, call *antisymmetric enveloping algebra* the quotient

$$A(D) := E(D)/J(D), \qquad J(D) := \langle u(a), v(a) \mid a\in D\rangle,$$

and *antisymmetric* a D-representation M which is a left module over $A(D)$ ($\Longleftrightarrow J(D)\cdot M = 0$), i.e. which satisfies the condition

$$(ARD)\qquad a\dashv m = 0 = m\vdash a, \qquad m\in M, \quad a\in D.$$

3.4 - Proposition. *For a dialgebra D, the category of symmetric D-representations, that of antisymmetric D-representations and the category of bimodules over D_{as} are all equivalent.*

Proof. Take a symmetric D-representation M and define an antisymmetric representation M' on the k-module M by setting $m \dashv' a := m \dashv a$, $m \vdash' a := 0$ and $a \vdash' m := a \vdash m$, $a \dashv' m := 0$. Similarly define a bimodule M_{as} over the associative algebra D_{as} by setting $m \cdot [a] := m \dashv a$ and $[a] \cdot m := a \vdash m$, where $[a]$ denotes the image of $a \in D$ in the projection $D \longrightarrow D_{as}$.

Both functors $M \mapsto M'$ and $M \mapsto M_{as}$ are faithful and full. For instance, if $f : M' \longrightarrow N'$ is a morphism of D-representations, where M' and N' are antisymmetric representations defined on symmetric representations M and N, then f is induced by the morphism $F : M \longrightarrow N$, $F(m) := f(m)$. Hence, with their inverse functors, they define equivalences of categories. **2**

3.5 - Examples.

1. The universal enveloping algebra $E(D)$ is itself, clearly, a representation of the dialgebra D, with actions given in (3.2) for $m \in E(D)$. From relations 5, 7, 10, 12, 13, 14 and 15 of (ED) follow immediately some additional properties of these actions (beside associativity), and in particular relations 7, 13, 15 and 10 imply that $J(D) \cdot I(D) = 0$, hence the ideal $I(D)$ of $E(D)$ is an antisymmetric representation of D.

2. $E(A)$ **for associative algebras.** For an associative algebra A, the classical enveloping algebra $A^e = A \otimes A^{op} \oplus A \oplus A^{op} \oplus k$ characterizes the category of bimodules over A. Regarding A as a dialgebra, the enveloping algebra $E(A) := E(A_{di})$ is generated by the set $\{x(a), u(a), y(a), v(a) | \, a \in A\}$ with the 15 relations (EA) obtained from (ED) by setting $\dashv = \vdash = \cdot$. In particular, these show that the maps $x, u : A \longrightarrow E(A)$ and $y, v : A^{op} \longrightarrow E(A)$ are morphisms of algebras. Then the unital morphism of algebras $A^e \longrightarrow E(A)$ defined by

$$A \ni a \mapsto x(a), \qquad A^{op} \ni b \mapsto y(b), \qquad A \otimes A^{op} \ni a \otimes b \mapsto x(a)y(b),$$

yields an inclusion of A^e in $E(A)$. Hence *the algebra A^e is isomorphic to the unital subalgebra of $E(A)$ generated by the elements $x(a), y(a)$ for any $a \in A$.*

If A has a unit 1, its classical enveloping algebra is $A^e = A \otimes A^{op}$, with unit $1 \otimes 1$. In $E(A)$, the elements $u(1)$ and $v(1)$ are also both side units, thus $x(1) = u(1) = y(1) = v(1) = 1_E$.

3. **Universal enveloping algebra of a Leibniz algebra.** Let $\mathfrak{g}$ be a Leibniz algebra. The *universal enveloping algebra* $UL(\mathfrak{g})$ is the quotient of the free associative unital algebra $T(\mathfrak{g}^l \oplus \mathfrak{g}^r)$ by the ideal corresponding to the three following relations:

$$(UL) \quad \begin{cases} r([a,b]) \overset{1}{=} -r(a)r(b) + r(b)r(a), \\ l([a,b]) \overset{2}{=} -l(a)r(b) + r(b)l(a), \\ l(a)[l(b) + r(b)] \overset{3}{=} 0, \end{cases} \qquad a, b \in D.$$

In [L-P] it is shown that the category of $\mathfrak{g}$-representations is equivalent to the category of left modules over $UL(\mathfrak{g})$. (Stricktly talking, in [L-P] representations of $\mathfrak{g}$ are *right* modules over the algebra $UL(\mathfrak{g})$, which is then slightely different from ours.) The assignement

$$r(a) \mapsto y(a) - x(a), \qquad l(a) \mapsto u(a) - v(a),$$

yields a morphism of algebras $UL(D_L) \longrightarrow E(D)$.

3.6 - Graded enveloping algebra. Given a dialgebra D, the tensor algebra $T(Dx \oplus Du \oplus Dy \oplus Dv)$ is endowed with a length-filtration, where the degree $\deg(e)$ of a homogeneus element

e is the length of the sequence of elements $x(a), u(b), y(c), v(d)$ componing e. In fact there is a canonical isomorphism of graded k-modules

$$T(Dx \oplus Du \oplus Dy \oplus Dv) \cong T(Dx) * T(Du) * T(Dy) * T(Dv),$$

where the symbol $*$ denotes the non-commutative tensor product[1]. This filtration induces on the quotient $E(D)$ a length-filtration given by the k-modules

$$F_k E(D) := \{e \in E(D) |\ \deg(e) \leq n\}, \qquad k \geq 0,$$

satisfying the standard properties $F_k E(D) \subset F_{k+1} E(D)$ for any $k \geq 0$, $\bigcup_{k \geq 0} F_k E(D) = E(D)$, $F_p E(D) \cdot F_q E(D) \subset F_{p+q} E(D)$ for any $p, q \geq 0$ and such that $F_0 E(D) = k$.

Denote by $GrE(D) = \bigoplus_{k \geq 0} Gr^k E(D)$ the graded algebra associated to this filtration, so $Gr^0 E(D) = k$ and $Gr^k E(D) = F_k E(D)/F_{k-1} E(D)$.

3.7 - Proposition. *Let D be a dialgebra with graded enveloping algebra $Gr^* E(D)$. Then,*

1. $Gr^1 E(D) \cong Dx \oplus Du \oplus Dy \oplus Dv$,
2. $Gr^2 E(D) \cong D^{\otimes 2} xy \oplus D^{\otimes 2} uv \oplus D^{\otimes 2} vu$,
3. $Gr^k E(D) = 0$ *for any* $k \geq 3$.

Proof. The assertion 1. is obvious.

2. An element of degree 2 in $E(D)$ is the class $[e]$ of an element of degree 2 of the tensor algebra $T(Dx \oplus Du \oplus Dy \oplus Dv)$, in the equivalence given by relations (ED). The element e must be of the form $e_1(a) \cdot e_2(b)$ for some elements $a, b \in D$, where each e_i, for $i = 1, 2$, is one of the four maps x, u, y, v. Among the 16 possible products $e_1(a) \cdot e_2(b)$ of this kind in the tensor algebra (for fixed $a, b \in D$), 7 products are identified with some others in the quotient algebra $E(D)$ because of relations (5), (7), (10), (12), (13), (14) and (15). Then, among the 9 distinct products which remain in $E(D)$, 6 products are equivalent to elements of degree 1 because of relations (1), (3), (6), (8), (9) and (11), yielding zero-classes in the quotient $F_2 E(D)/F_1 E(D)$. Hence in $F_2 E(D)/F_1 E(D)$ there remain only three distinct products, which can be chosen to be $x(a)y(b)$, $u(a)v(b)$ and $v(a)u(b)$ for any $a, b \in D$.

3. It suffices to show the identity $F_k E(D) = F_{k+1} E(D)$ for $k = 2$. Since the elements of degree 2 in $E(D)$ are of the form descrived above, it is easy to see by direct computations, using relations (ED), that any element of degree 2 multiplied from left or right for an element of degree 1 yields an element still of degree 2. For instance, the element $x(a)y(b)$ multiplied for $u(c)$ from left yields an element of degree 2 by relation (8). □

3.8 - Corollary (Poincaré-Birchoff-Witt Isomorphism). *Let V denote the underlying k-module of a dialgebra D. Define on V a structure of abelian dialgebra, by setting $a \star b := 0$ for any $a, b \in V$. Then there is an isomorphism of associative unital algebras*

$$GrE(D) \cong E(V).$$

[1]The symbol $*$ is defined on two k-modules V and W by $V * W := V \otimes W \oplus W \otimes V$, [L4]. It can be extended to graded k-modules $T(V)$ and $T(W)$ as the graded k-module with homogeneous components

$$(T(V) * T(W))^{(n)} = \bigoplus_{\underline{\lambda} \vdash n} V^{\otimes \lambda_1} \otimes W^{\otimes \lambda_2} \otimes V^{\otimes \lambda_3} \otimes \cdots \otimes W^{\otimes \lambda_k},$$

where the direct sum is taken over all partitions $\underline{\lambda}$ of n (ordered sets of integers $(\lambda_1, ..., \lambda_k)$ such that $\sum_i \lambda_i = n$).

4 Dialgebra homology and corepresentations

In this section we introduce the dialgebra homology with coefficients as the dual theory of the cohomology defined in section 2.

4.1 - Corepresentations. Let D be a dialgebra with universal enveloping algebra $E(D)$. By definition, a *corepresentation* N of D is a right module over $E(D)$. Set

$$\begin{array}{ll} a \Vdash n := n \cdot x(a), & n \dashV a := n \cdot y(a), \\ a \dashV n := n \cdot u(a), & n \Vdash a := n \cdot v(a), \end{array} \qquad n \in N, \quad a \in D.$$

Then the k-module N is equipped with two right actions $\dashV, \Vdash \colon N \otimes D \longrightarrow N$ and two left actions $\dashV, \Vdash \colon D \otimes N \longrightarrow N$ satisfying the following 15 axioms:

$$(CD) \quad \begin{cases} n \dashV (a \dashv b) \overset{1}{=} (n \dashV b) \dashV a \overset{2}{=} n \dashV (a \vdash b), \\ (n \dashV a) \Vdash b \overset{3}{=} n \Vdash (b \dashv a), \\ (n \Vdash a) \Vdash b \overset{4}{=} n \Vdash (b \vdash a) \overset{5}{=} (n \Vdash a) \dashV b, \\ (a \dashV n) \dashV b \overset{6}{=} a \dashV (n \dashV b) \overset{7}{=} (a \dashV n) \Vdash b, \\ (a \Vdash n) \dashV b \overset{8}{=} a \Vdash (n \dashV b), \\ a \Vdash (n \Vdash b) \overset{9}{=} (a \Vdash n) \Vdash b \overset{10}{=} a \dashV (n \Vdash b), \\ a \dashV (b \dashV n) \overset{11}{=} (b \dashv a) \dashV n \overset{12}{=} a \Vdash (b \dashV n), \\ (a \vdash b) \dashV n \overset{13}{=} b \dashV (a \Vdash n), \\ (a \vdash b) \Vdash n \overset{14}{=} b \Vdash (a \Vdash n) \overset{15}{=} (a \dashv b) \Vdash n, \end{cases} \qquad n \in N, \quad a, b \in D.$$

If D has a bar-unit 1, a corepresentation N is called *unital* if $1 \Vdash n = n = n \dashV 1$ for all $n \in N$.

Notice that, unlike for representations, a dialgebra D is *not* a corepresentation of itself!

If D_L is the canonical Leibniz algebra of D, on the k-module N we obtain a corepresentation N_L of D_L, as defined in [L-P], by setting

$$[\![n, a]\!] := a \dashV n - n \Vdash a \qquad \text{and} \qquad [\![a, n]\!] := n \dashV a - a \Vdash n, \qquad n \in N, \quad a \in D.$$

4.2 - Symmetric and antisymmetric corepresentations. Let D be a dialgebra. A D-corepresentation N is called *symmetric* if it is a right module over the symmetric enveloping algebra $S(D)$. This is equivalent to require that the actions of $E(D)$ on N satisfy the conditions

$$(SCD) \quad \begin{cases} n \dashV a - n \Vdash a = 0, \\ a \dashV n - a \Vdash n = 0, \end{cases} \qquad n \in N, \quad a \in D.$$

The corepresentation N is called *antisymmetric* if it is a right module over the antisymmetric enveloping algebra $A(D)$, i.e. if it satisfies

$$(ACD) \qquad a \dashV n = 0 = n \Vdash a, \qquad n \in N, \quad a \in D.$$

If N is a symmetric corepresentation of D (resp. antysimmetric), then N_L is a symmetric corepresentation of D_L (resp. antisymmetric).

4.3 - Examples.

1. Given a dialgebra D over k, any k-module N becomes a D-corepresentation, called *trivial*, by setting $n \star a := 0 =: a \star n$ for any $n \in N$ and $a \in D$. (Here $\star$ denotes both symbols $\dashV$ and $\Vdash$.) If D is bar-unital, the trivial corepresentation can not be unital, unless $N = 0$.

2. Given a dialgebra D with a corepresentation N, the k-module $\mathrm{M}_r(N)$ of matrices with entries in N is a corepresentation of the dialgebra $\mathrm{M}_r(D)$, with 'row $\times$ column' actions.

3. The universal enveloping algebra $E(D)$ is itself, clearly, a corepresentation of the dialgebra D, with actions given in (4.1) for $n \in E(D)$. Moreover, relations 7, 15, 13, 10 of (ED) imply that $J(D) \cdot I(D) = 0$, hence the ideal $J(D)$ of $E(D)$ is a symmetric D-corepresentation.

4. **Isotypical decomposition.** Given a corepresentation N of a dialgebra D, the quotient

$$S(N) := N/NI(D), \qquad NI(D) := \{n \cdot e \in N \mid n \in N, e \in I(D)\},$$

is a symmetric corepresentation, since $S(N) \cdot I(D) = 0$. Similarly, the quotient of N

$$A(N) := N/NJ(D), \qquad NJ(D) := \{n \cdot e \in N \mid n \in N, e \in J(D)\},$$

is an antisymmetric corepresentation, because $A(N) \cdot J(D) = 0$. Moreover $NJ(D)$ is itself a symmetric corepresentation of D, hence N can be decomposed into the direct sum

$$N \cong A(N) \oplus NJ(D),$$

where $A(N)$ is antisymmetric and $NJ(D)$ is symmetric.

5. **Opposite corepresentations.** Let D be a dialgebra. Given a D-representation M, we can define two D-corepresentations on the k-module M. The first one, with actions

$$m \dashv\!\mid a := a \vdash m =: m \Vdash a, \qquad a \dashv\!\mid m := m \dashv a =: a \Vdash m,$$

is by definition a symmetric corepresentation of D, and the second one, with actions

$$m \dashv\!\mid a := a \vdash m, \qquad m \Vdash a := 0, \qquad a \Vdash m := m \dashv a, \qquad a \dashv\!\mid m := 0,$$

is antisymmetric. We call them *symmetric* and *antisymmetric opposite corepresentation* of M respectively, and denote them both by M^{op}. In particular, since D is a representation over itself, we can consider the symmetric or antisymmetric corepresentation D^{op}.

This procedure can be inverted. Given a corepresentation N, analogue formulas give the *symmetric* and *antisymmetric opposite representation* of N, denoted both by N^{op}.

6. Let D be a dialgebra, and let D^{op} be the symmetric corepresentation on D. Then D_L^{op} is a corepresentation of D_L, with actions $[\![a, b]\!] = b \dashv\!\mid a - a \Vdash b = a \dashv b - b \vdash a = [a, b]$ and $[\![b, a]\!] = a \dashv\!\mid b - b \Vdash a = b \vdash a - a \dashv b = -[a, b]$, for any $a \in D_L^{op}$ and $b \in D_L$. It is clearly symmetric.

4.4 - Proposition. *The category of symmetric (or antisymmetric) representations of a dialgebra D, that of symmetric (or antisymmetric) corepresentations of D and the category of bimodules over the associative algebra D_{as} are all equivalent.*

Proof. After (3.4), we only need to show one of these equivalences, for instance that from antisymmetric representations to antisymmetric corepresentations. Take a morphism of corepresentations $f : M^{op} \longrightarrow M'^{op}$, where M^{op} and M'^{op} are the opposite of two representations. Then the k-linear map $f : M \longrightarrow M'$, on the underlying k-modules, is a morphism of representations. In fact,

$$f(m \dashv a) = f(a \Vdash m) = a \Vdash f(m) = f(m) \dashv a, \qquad m \in M, \quad a \in D,$$

and similarly $f(m \vdash a) = 0 = f(a \dashv m)$ and $f(a \vdash m) = a \vdash f(m)$. Hence the functor $M \mapsto M^{op}$ is full. Now take two representations M and M' such that their opposite corepresentations are isomorphic, $M^{op} \cong M'^{op}$. Clearly they are isomorphic as k-modules, $M \cong M'$, and also as D-representations, in fact,

$$m \dashv a = a \Vdash m = a \Vdash' m = m \dashv' a, \qquad m \in M, \quad a \in D,$$

and similarly $m \vdash a = 0 = m \vdash' a$, $a \dashv m = 0 = a \dashv' m$, and $a \vdash m = a \vdash' m$. Hence the functor $M \mapsto M^{op}$ is faithful. Finally, by definition, the functor $N \mapsto N^{op}$ from an antisymmetric corepresentation to its opposite antisymmetric representation is its inverse. $\square$

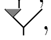
4.5 - Dialgebra complex. Let D be a dialgebra over k and N be a corepresentation of D. For any $n \geq 0$, define the *module of n-chains of D with coefficients in N* as

$$CY_n(D, N) := k[Y_n] \otimes N \otimes D^{\otimes n},$$

and the boundary operator $d : CY_n(D, N) \longrightarrow CY_{n-1}(D, N)$ as the k-linear map $d = \sum_{i=0}^{n} (-1)^i d_i$ given by means of the face maps $d_i(y \otimes \underline{a}) = d_i(y) \otimes d_i^y(\underline{a})$, where the faces d_i on the tree y are defined in (2.3) and for any $\underline{a} = (a_0, a_1, ..., a_n) \in N \otimes D^{\otimes n}$ the terms

$$d_i^y(a_0, a_1, ..., a_n) := \begin{cases} (a_1 \circ_0^y a_0, a_2, ..., a_n), & \text{for } i = 0, \\ (a_0, a_1, ..., a_i \circ_i^y a_{i+1}, ..., a_n) & \text{for } i = 1, ..., n-1, \\ (a_0 \circ_n^y a_n, a_1, ..., a_{n-1}), & \text{for } i = n, \end{cases}$$

contain the operations $\circ_i^y$ defined in (2.3). (Notice the side on which the elements a_1 and a_n of D act on the element a_0 of N.) In particular, the operations $\circ_0^y$ and $\circ_n^y$ yield

$a_1 \circ_0^y a_0 := a_1 \mathrel{\dashv\!\!\mid} a_0$, if y has the shape ,
$a_1 \circ_0^y a_0 := a_1 \Vdash a_0$, if y has the shape ,
$a_0 \circ_n^y a_n := a_0 \Vdash a_n$, if y has the shape ,
$a_0 \circ_n^y a_n := a_0 \mathrel{\dashv\!\!\mid} a_n$, if y has the shape .

When $N = D^{op}$ is the symmetric corepresentation on D, the module of n-chains is

$$CY_n(D, D^{op}) = k[Y_n] \otimes D^{\otimes n+1},$$

and the face maps d_0 and d_n take the simple form

$$d_0(y \otimes (a_0, a_1, ..., a_n)) = d_0(y) \otimes (a_0 \dashv a_1, a_2, ..., a_n),$$
$$d_n(y \otimes (a_0, a_1, ..., a_n)) = d_n(y) \otimes (a_n \vdash a_0, a_1, ..., a_{n-1}),$$

for any n-tree y and $(a_0, a_1, ..., a_n) \in D^{\otimes n+1}$.

In order to prove that $(CY_*(D, N), d)$ is a chain complex, it suffices to show that the chain modules $CY_n(D, N)$ are dual to the cochain modules $CY^n(D, M)$, in the category of modules over $E(D)$, and that the faces d_i and δ^i correspond each other in this duality. To prove it, we introduce the suitable 'mean term' of the duality.

4.6 - Cotangent complex of a dialgebra. We call *module of cotangent n-chains* of a dialgebra D the module of n-chains of D with coefficients in the universal enveloping algebra $E(D)$,

$$EY_n(D) := CY_n(D, E(D)) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}.$$

The first and last face maps, in this case, take the following form:

$$d_0(y \otimes (e, a_1, ..., a_n)) = d_0(y) \otimes \begin{cases} (e \cdot u(a_1), a_2, ..., a_n), & \text{if } y \text{ has the shape}, \\ (e \cdot x(a_1), a_2, ..., a_n), & \text{if } y \text{ has the shape}, \end{cases}$$

and similarly

$$d_n(y \otimes (e, a_1, ..., a_n)) = d_n(y) \otimes \begin{cases} (e \cdot v(a_n), a_0, ..., a_{n-1}), & \text{if } y \text{ has the shape}, \\ (e \cdot y(a_n), a_0, ..., a_{n-1}), & \text{if } y \text{ has the shape}, \end{cases}$$

for any n-tree y, any element $e \in E(D)$ and $a_i \in D$ for $i = 1, ..., n$. In particular, the first boundary operator $d : E(D) \otimes D \longrightarrow E(D)$ is explicitely defined as

$$d(e \otimes a) = e \cdot (u(a) - v(a)), \qquad e \in E(D), \qquad a \in D.$$

Remark that the cotangent complex in the context of associative algebras is the bar complex shifted by 1, while for commutative algebras it was defined by D. Quillen in [Q].

4.7 - Lemma. *Let D be a dialgebra with universal enveloping algebra $E(D)$.*

(i) For any D-representation M, the isomorphism of cochain modules

$$CY^n(D, M) \cong Hom_{E(D)}(EY_n(D), M)$$

commutes with the coboundary δ on $CY^*(D, M)$ and the d^* induced on $Hom_{E(D)}(EY_*(D), M)$ by the boundary operator d on $EY_*(D)$. Therefore $(EY_*(D), d)$ is a chain complex.

(ii) For any corepresentation N of D the isomorphism of chain modules

$$CY_*(D, N) \cong N \otimes_{E(D)} EY_*(D)$$

commutes with the boundaries. Hence $(CY_*(D, N), d)$ is also a chain complex.

Proof. (i) For any $n \geq 0$, the module of cotangent n-chains $EY_n(D) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}$ is a left module over $E(D)$, with action given by the multiplication in the algebra $E(D)$. Therefore, in the dual space $Hom_{E(D)}(EY_n(D), M)$ of left $E(D)$-modules morphisms from $EY_n(D)$ to a D-representation M, it suffices to consider the value of any map f on the unit 1_E of $E(D)$, since

$$f(y \otimes e \otimes \underline{a}) = e \cdot f(y \otimes 1_E \otimes \underline{a})$$

for any $e \in E(D)$. Hence the canonical isomorphism of n-cochain modules

$$CY^n(D, M) = \mathrm{Hom}_k(k[Y_n] \otimes D^{\otimes n}, M) \cong Hom_{E(D)}(EY_n(D), M) = EY_n(D)^*$$

associates a k-linear map $f : k[Y_n] \otimes D^{\otimes n} \longrightarrow M$ to its $E(D)$-linear extension $\tilde{f}(y \otimes e \otimes \underline{a}) := e \cdot f(y \otimes \underline{a})$, and conversely an $E(D)$-linear map $g : k[Y_n] \otimes E(D) \otimes D^{\otimes n} \longrightarrow M$ to its restriction on $\{1_E\} \subset E(D)$, $\tilde{g}(y \otimes \underline{a}) := g(y \otimes 1_E \otimes \underline{a})$. Then the maps $d_i^* : Hom_{E(D)}(EY_n(D), M) \longrightarrow Hom_{E(D)}(EY_{n+1}(D), M)$ dual to the faces d_i defined above satisfy

$$(d_i^* f)(y \otimes 1_E \otimes \underline{a}) := f(d_i(y \otimes 1_E \otimes \underline{a})) = f(d_i(y) \otimes 1_E \otimes d_i(\underline{a})) = (\delta^i f)(y \otimes \underline{a}),$$

for any $i = 1, ..., n$, and similarly

$$\begin{aligned}(d_0^* f)(y \otimes 1_E \otimes \underline{a}) &= f(d_0(y) \otimes a_0 \circ_0^y 1_E \otimes \underline{a}') = l(a_1) \cdot f(d_0(y) \otimes \underline{a}') \\ &= a_0 \circ_0^y f(d_0(y) \otimes \underline{a}') = (\delta^0 f)(y \otimes \underline{a}),\end{aligned}$$

where $l(a_1) = x(a_1)$ or $u(a_1)$, and finally

$$\begin{aligned}(d_{n+1}^* f)(y \otimes 1_E \otimes \underline{a}) &= f(d_{n+1}(y) \otimes 1_E \circ_{n+1}^y a_{n+1} \otimes \underline{a}'') = r(a_{n+1}) \cdot f(d_{n+1}(y) \otimes \underline{a}'') \\ &= f(d_{n+1}(y) \otimes \underline{a}'') \circ_{n+1}^y a_{n+1} = (\delta^{n+1} f)(y \otimes \underline{a}),\end{aligned}$$

where $r(a_1) = y(a_{n+1})$ or $v(a_1)$. It follows that the faces d_i on $EY_*(D)$ satisfy the dual simplicial relations $d_i d_j = d_{j-1} d_i$ for any $i < j$, and $(EY_*(D), d)$ is a chain complex.

(ii) This follows, by applying the isomorphism of chain modules

$$CY_*(D, N) = k[Y_n] \otimes N \otimes D^{\otimes n} \cong k[Y_n] \otimes N \otimes_{E(D)} E(D) \otimes D^{\otimes n},$$

with straightforward computations on the faces. □

4.8 - Definition (dialgebra homology). Let D be a dialgebra and N a corepresentation of D. We call *dialgebra homology of D with coefficients in N* the homology of the chain complex defined in (4.5),

$$HY_n(D, N) := H_n(CY_*(D, N), d), \qquad n \geq 0.$$

We call *dialgebra homology of D with coefficients in itself* the homology

$$HHY_*(D) := HY_*(D, D^{op})$$

with coefficients in the symmetric corepresentation D^{op}. When A is an associative algebra, and M is an A-bimodule, we simply denote by

$$HY_*(A, M) := HY_*(A_{di}, M^{op})$$

the homology of A regarded as a dialgebra, where M^{op} is the symmetric corepresentation on M.

4.9 - Elementary computations. Given a dialgebra D with a corepresentation N, the first boundary of D in N is the k-linear map $d : N \otimes D \longrightarrow N$, $d(n, a) := a \dashv\!\!| \, n - n \, |\!\!\vdash a$. Thus $HY_0(D, M)$ is the module of coinvariants of N by the Leibniz algebra D_L,

$$HY_0(D, N) = N / [\![N, D]\!],$$

where $[\![N, D]\!] := \{ [\![n, a]\!] = a \dashv\!\!| \, n - n \, |\!\!\vdash a \mid n \in N, a \in D\}$. In particular, when $N = D^{op}$, by definition of the actions of D on D^{op} we recover the Leibniz bracket $d(a, b) = b \dashv\!\!| \, a - a \, |\!\!\vdash b = a \dashv b - b \vdash a = [a, b]$ for any $a, b \in D$. Hence $[\![D^{op}, D]\!] = [D, D]$ and

$$HHY_0(D) = D/[D, D].$$

When $N = E(D)$ is the universal enveloping algebra of D, and D has a bar-unit 1, we can define a natural map $\pi : E(D) \longrightarrow D$ of left $E(D)$-modules as the $E(D)$-linear extension of the map which sends the unit of $E(D)$ into the bar-unit of D,

$$\pi : E(D) \longrightarrow D, \quad e \mapsto e \cdot 1,$$

where $e \cdot 1$ is the left action of $E(D)$ on D. The map π is then an augmentation for the cotangent complex, that is, $\pi \circ d = 0$, where $d : E(D) \otimes D \longrightarrow E(D)$ is the first boundary. In fact,

$$\pi \circ d(e \otimes a) = \pi(e \cdot (u(a) - v(a))) = e \cdot (u(a) - v(a)) \cdot 1 = e \cdot (a \dashv 1 - 1 \vdash a) = 0,$$

for any $e \in E(D)$ and $a \in D$. Hence

$$HY_0(D, E(D)) = M(D) := E(D)/\langle u(a) - v(a) |\ a \in D\rangle \cong \ker\pi.$$

4.10 - Bar-homology of dialgebras. When $N = k$ is the trivial D-corepresentation, the faces d_0 and d_n become zero on the module of n-chains of D. The boundary operator d then coincides with the boundary $d' = \sum_{i=1}^{n-1} (-1)^i d_i$ defined in [L5], and we recover the bar homology of D,

$$HY_*(D, k) = HY_*(D).$$

In particular, clearly, $HY_0(D, k) \cong k$.

For associative algebras it is well known that the Hochschild bar complex yields zero on homology, when the algebra has a unit (Proposition (1.1.12) in [L1]). The same holds for dialgebras, when they are equipped with a bar-unit.

4.11 - Theorem. *Let D be a dialgebra with bar-unit. Then the bar complex of D is acyclic, i.e.*

$$HY_n(D) = 0, \qquad n \geq 1.$$

Proof. Consider the identity and the zero maps on the chain complex $CY_*(D)$. If the dialgebra D has a bar-unit 1, we can define a homotopy between these two maps by setting

$$h : k[Y_n] \otimes D^{\otimes n} \longrightarrow k[Y_{n+1}] \otimes D^{\otimes n+1}, \qquad h(y \otimes \underline{a}) := s_0(y) \otimes (1, \underline{a}),$$

where $\underline{a} = (a_1, ..., a_n)$ is a generic element of $D^{\otimes n}$, and $s_0(y)$ is the $(n+1)$-tree obtained by bifurcating the 0^{th} leaf of y, i.e. by replacing the 0^{th} leaf $\mid$ with the branch $\curlyvee$. In order to show that

$$d'h + hd' = id,$$

we only need to verify that $d_1 h = id$ and $d_i h = h d_{i-1}$ for any $i = 2, ..., n+1$. In fact, for the first equality we have

$$d_1 h(y \otimes \underline{a}) = d_1 s_0(y) \otimes (1 \vdash a_1, ..., a_n) = y \otimes \underline{a},$$

because clearly $d_1 s_0$ deletes a leaf which has just been added, and for the second equality we have

$$\begin{aligned} d_i h(y \otimes \underline{a}) &= d_i s_0(y) \otimes (1, a_1, ..., a_{i-1} \circ_i^{s_0(y)} a_i, ..., a_n) \\ &= s_0 d_{i-1}(y) \otimes (1, a_1, ..., a_{i-1} \circ_{i-1}^{y} a_i, ..., a_n) = h d_{i-1}(y \otimes \underline{a}), \end{aligned}$$

if we observe that the i^{th} leaf of $s_0(y)$ is the leaf number $i-1$ of y, and that the two operators $d_i s_0$ and $s_0 d_{i-1}$ produce the same modification. □

The hypothesis that D has a bar-unit can be relieved to the assumption that there is an element $1 \in D$ such that $1 \vdash a = a$ for all $a \in D$. Dually, we could require that $a \dashv 1 = a$ and then use the homotopy $h(y \otimes \underline{a}) := s_n(y) \otimes (\underline{a}, 1)$, where s_n bifurcates the n^{th} leaf of y.

5 Dialgebra (co)homology as derived functor

In this section we prove our main theorem.

5.1 - Theorem. *Let D be a dialgebra and $E(D)$ its universal enveloping algebra. For any corepresentation N of D, there is an isomorphism*

$$HY_n(D,N) \cong \mathrm{Tor}_n^{E(D)}(M(D),N), \qquad n \geq 0,$$

where the left $E(D)$-module $M(D) := E(D)/\langle u-v\rangle$ is the cokernel of the first boundary operator.

Dually, for any representation M of D, there is an isomorphism

$$HY^n(D,M) \cong \mathrm{Ext}^n_{E(D)}(M,M(D)), \qquad n \geq 0.$$

Proof. Let us consider the cotangent chain complex $EY_n(D) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}$. If N is any D-corepresentation, by (4.7) we have an isomorphism of complexes $CY_*(D,N) \cong N \otimes_{E(D)} EY_*(D)$. The chain modules $EY_n(D)$ are free over $E(D)$, hence projective. Then, in order to prove the Tor interpretation for dialgebra homology, we only need to show that $EY_*(D)$ is a resolution of $M(D)$. To this proof, which follows from corollary (5.10), is devoted the remaining part of this section. The second statement follows from the duality between homology and cohomology. □

5.2 - Definition (dialgebra chain bicomplex). Let D be a dialgebra over a field k, and N a D-corepresentation. For any $p, q \geq 0$, consider the set $Y_{p,q}$ of planar binary trees with p leaves oriented like \ and q leaves oriented like / Define a k-linear map $d^h : CY_{p,q}(D,N) \longrightarrow CY_{p-1,q}(D,N)$, $d^h := \sum_{i=0}^{p+q+1}(-1)^i d_i^h$, by means of the face maps

$$d_i^h(y \otimes \underline{a}) := d_i^h(y) \otimes d_i^y(\underline{a}),$$

where $d_i^h(y)$ is the horizontal face acting on the tree, defined as $d_i(y)$, if $d_i(y) \in k[Y_{p-1,q}]$, and 0 otherwise, and the term $d_i^y(\underline{a})$ is defined in (4.5). Similarly, using the vertical faces on trees, we can introduce a k-linear map $d^v : CY_{p,q}(D,N) \longrightarrow CY_{p,q-1}(D,N)$.

5.3 - Lemma. *The family $(CY_{p,q}(D,N), d^h, d^v)$ forms a bicomplex, whose total complex is the shifted dialgebra complex $CY_{*-1}(D,N)$.*

Proof. By definition, the orientation of the faces on $CY_{p,q}(D,N)$ just depends on the orientation of the face maps in the bicomplex of trees. So the identities $d^hd^h = 0 = d^vd^v$ and $d^vd^h + d^hd^v = 0$, follow immediately from lemma (4.7) and the analogue result on the bicomplex of trees which is proved in [F4]. Finally, since

$$CY_n(D,N) = \bigoplus_{p+q+1=n} CY_{p,q}(D,N)$$

and $d = d^h + d^v$, it is evident that $CY_{*-1}(D,N) = \mathrm{Tot}(CY_{*,*}(D,N)$. □

As a consequence, there is a spectral sequence abutting to the dialgebra homology.

5.4 - Theorem. *Let D be a dialgebra and N a D-corepresentation. There is a spectral sequence $E^2_{pq} \Longrightarrow HY_{p+q+1}(D,N)$ which converges to the dialgebra homology of D, with*

$$\begin{aligned} E^1_{pq} &:= H_q(CY_{p,*}(D,N), d^v),\\ E^2_{pq} &:= H_p(E^1_{*q}, (d^h)_*) = H_pH_q(CY_{*,*}(D,N)). \end{aligned}$$

5.5 - Vertical towers. For any p-tree y, let $T_*(y) \subset k[Y_{p,*}]$ be the vertical tower defined in [F4]. Recall that the tower $T_*(y)$ is constructed on the base tree associated to y (which is the $(p,*)$-tree $\tilde{y}$ obtained by bifurcating all the leaves of y), by adding /-leaves in all possible distinct ways. In [F4] it is shown that the tower $T_*(y)$, on a p-tree y, is a multicomplex with dimension $d = 2p+1$ in the vertical complex $k[Y_{p,*}]$, and moreover that $k[Y_{p,*}]$ can be decomposed as the direct sum of the towers $T_*(y)$, for $y \in Y_p$.

Now consider a dialgebra D and a D-corepresentation N. For any p-tree y, we define a family of dialgebra chain submodules

$$CT^y_m(D,N) := T_m(y) \otimes N \otimes D^{\otimes p+q_y+m}, \qquad m \geq 0$$

where q_y denotes the number of /-leaves of y. It is easy to see that the family $CT^y_*(D,N)$ is a multi-complex with dimension $d = 2p+1$ in the vertical complex $CY_{p,*}(D,N)$.

5.6 - Lemma. *Let D be a dialgebra and N a D-corepresentation. For any $p \geq 0$, the vertical complex $(CY_{p,*}(D,N), d^v)$ decomposes as a direct sum of sub-complexes,*

$$CY_{p,*}(D,N) = \bigoplus_{y \in Y_p} CT^y_{*+q_y}(D,N).$$

Consequently, the E^1-plane of the spectral sequence (5.4) is

$$E^1_{pq} = \bigoplus_{y \in Y_p} H_q(CT^y_{*+q_y}(D,N), d^v).$$

Proof. Applying the definition of the towers one sees that this result does not involve the dialgebra structure of D. Hence it follows from the analogue proposition on towers of trees proved in [F4]. □

5.7 - Proposition. *Let D be an abelian dialgebra, and $E(D)$ its universal enveloping algebra. Then $HY_0(D,E(D)) \cong E(D)/\langle u-v\rangle$ and*

$$HY_n(D,E(D)) = 0, \qquad n \geq 1.$$

Proof. Consider the bicomplex $EY_{p,q}(D) = k[Y_{p,q}] \otimes E(D) \otimes D^{\otimes p+q+1}$ which gives rise to the spectral sequence $E^2_{pq} \Longrightarrow HY_{p+q+1}(D,E(D))$. Since D is an abelian dialgebra, the only non-zero faces are the first and the last one, hence $d = d_0 + (-1)^n d_n$. They act on a generic element of $EY_{p,q}(D)$ as

$$d^y_0(e \otimes \underline{a}) = \begin{cases} e \cdot x(a_1) \otimes (a_2,..,a_n)\\ e \cdot u(a_1) \otimes (a_2,..,a_n)\end{cases} \qquad \text{and} \qquad d^y_n(e \otimes \underline{a}) = \begin{cases} e \cdot y(a_n) \otimes (a_1,..,a_{n-1})\\ e \cdot v(a_n) \otimes (a_1,..,a_{n-1})\end{cases}$$

dependently on the shape of the first and last leaves of the tree y (here $n = p+q+1$). The horizontal and vertical boundary operators are such that $d = d^h + d^v$, hence the two non-zero faces are distributed in d^h and d^v in the four possible combinations.

1) Let us compute the E^1-plane, $E^1_{pq} = H_q(EY_{p,*}(D), d^v) = \bigoplus_{y\in Y_p} H_q(ET^y_{p,*}(D), d^v)$.

• For $p = 0$, both non-zero faces are obviously vertical, and because of the shape of the comb trees, they always act as multiplications by elements $x(a_1)$ and $v(a_n)$, hence the vertical tower for $p = 0$ is the complex

$$\cdots \longrightarrow E(D)\otimes D^{\otimes n} \xrightarrow{x\pm v} E(D)\otimes D^{\otimes n-1} \xrightarrow{x\mp v} \cdots \xrightarrow{x-v} E(D)\otimes D^{\otimes 2} \xrightarrow{x+v} E(D)\otimes D.$$

For $q = 0$, we get

$$E^1_{00} = H_0(EY_{0,*}(D), d^v) \cong \frac{E(D)\otimes D}{\langle x(a)\otimes b + v(b)\otimes a\rangle}.$$

For $q = 1$, the kernel of the map $e\otimes(a,b) \mapsto ex(a)\otimes b + ev(b)\otimes a$ is precisely the left $E(D)$-module of elements $\langle x(a)\otimes(b,c) - v(c)\otimes(a,b)\rangle$, hence

$$E^1_{01} = H_1(EY_{0,*}(D), d^v) = 0.$$

Furthermore, for $q > 1$, we get

$$E^1_{0q} \cong E^1_{01}\otimes D^{\otimes q-1} = 0.$$

• For $p \geq 1$, observe that on each copy $E(D)\otimes D^{\otimes p+q+1}$, labeled by a (p,q)-tree y, the two non-zero faces are necessarily distributed one in d^h and the other one in d^v. Hence, when we consider each $(2p+1)$-dimensional vertical tower, on each copy $E(D)\otimes D^{\otimes p+q+1}$ the vertical boundary is just the right multiplication on $E(D)$ by an element among $x(a_1)$, $u(a_1)$, $y(a_n)$, or $v(a_n)$. Let us give more details for $p = 1$.

In the 3-dimensional vertical tower, each $(1,q)$-tree labels the chain module $E(D)\otimes D^{\otimes q+2}$, and the boundaries, in the three directions I, II and III, are given as multiplications by the terms $x(a_1)$, $(-1)^n y(a_n)$ and $(-1)^n v(a_n)$, respectively. The total homology can be found with two spectral sequences [2]. The first one, $A^2_{rs} := H^{II}_r H^I_s \Longrightarrow H^{I,II}_{r+s}(EY_{1,*}(D))$, which abuts to the total homology of the bicomplex determined by the plane I, II, has stable plane $A^\infty_{rs} = A^2_{rs}$ and homology $H^{I,II}_z = \bigoplus_{r+s=z} A^2_{rs}$. The second one, $B^2_{tz} := H^{III}_t H^{I,II}_z \Longrightarrow H^{I,II,III}_{t+z}(EY_{1,*}(D), d^v)$, which converges to the total homology of the 3-complex, has stable plane $B^\infty_{tz} = B^2_{tz}$ described in fig. (5.1). Hence the higher terms E^1_{1q} are direct sums of copies of the terms $\frac{Eu}{Eyu}\otimes D^{\otimes q+2}$, $Evu\otimes D^{\otimes q+2}$, $\frac{Ev}{Exv}\otimes D^{\otimes q+2}$, $\frac{E}{E(x,y,v)}\otimes D^{\otimes q+2}$.

• For $p \geq 2$ we find a similar result, using $(2p+1)$ spectral sequences on each vertical tower.

2) When we compute the term $E^2_{pq} = H_p(E^1_{*q}, (d^h)_*) = H_p H_q(EY_{*,*}(D))$, we have a similar situation. Each (p,q)-tree y labels now a quotient or a sub-module of $E(D)\otimes D^{\otimes p+q+1}$. The induced boundary $(d^h)^*$ yields zero on the horizontal homolgy, except in degree $(0,0)$, hence

$$HY_n(D, E(D)) = 0, \qquad n \geq 2.$$

In degree $(0,0)$ we have

$$E^2_{00} = \frac{(E(D)\otimes D)/\langle x(a)\otimes b + v(b)\otimes a\rangle}{\langle u(a)\otimes b + y(b)\otimes a\rangle}.$$

[2] **Total homology of a multicomplex.** Let us discuss briefly how to compute the homology of a multicomplex with dimension $d > 1$. Denote by I, II, III, ..., the d directions of the boundary operators. Any two such directions determine a bicomplex whose total homology can be computed with a spectral sequence. Hence we have $d-1$ spectral sequences

$$H^I_r H^{II}_s \Longrightarrow H^{I,II}_{r+s}, \qquad H^{III}_t H^{I,II}_z \Longrightarrow H^{I,II,III}_{t+z}, \qquad \ldots,$$

whose final abutment is the total homology of the multicomplex. Each abutment is computed, as usual, with the help of the short exact sequences determined by its filtration.

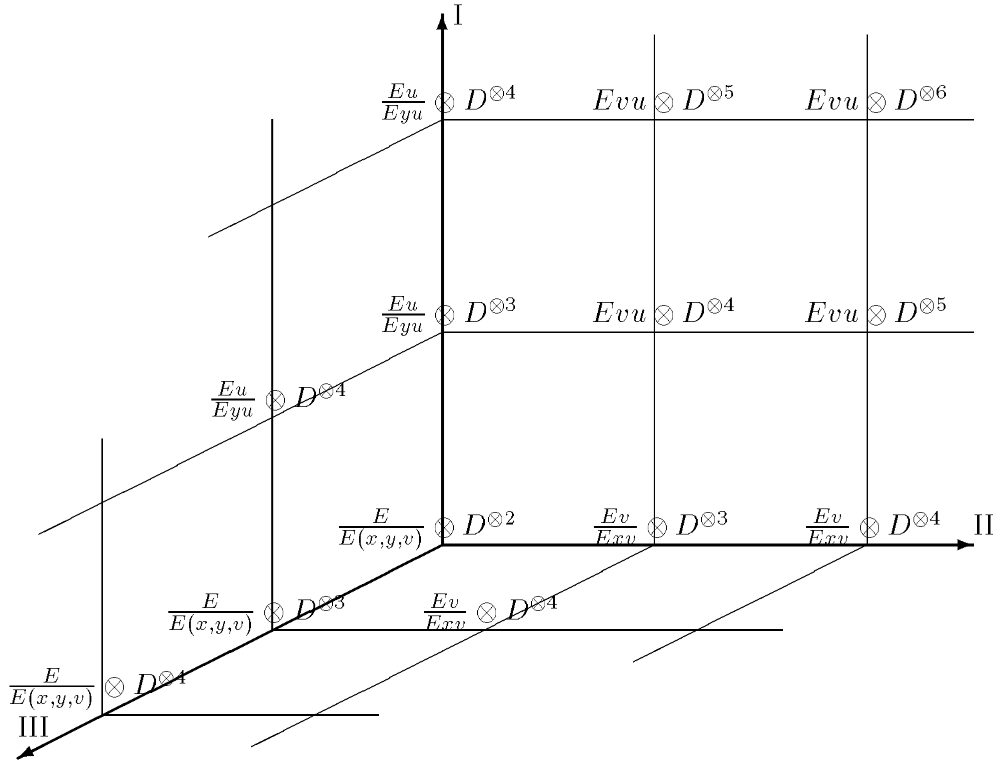


Figure 5.1: homology of the vertical tower with base $\curlyvee$.

Notations:

$\frac{E}{E(x,y,v)} = E(D)/\langle ex(a), ey(a), ey(a) \mid e \in E(D), a \in D\rangle$,
$\frac{Eu}{Eyu} = \frac{\{eu(a) \mid e \in E(D), a \in D\}}{\{ey(a)u(b) \mid e \in E(D), a, b \in D\}}$,
$Evu = \{ev(a)u(b) \mid e \in E(D), a, b \in D\}$,

3) Finally we observe the only term of the dialgebra chain complex which is not covered by the bicomplex, the boundary operator $E(D) \otimes D \longrightarrow E(D), e \otimes a \mapsto e(u(a) - v(a))$. The induced map $E^2_{00} \longrightarrow E(D)$ has zero kernel, hence

$$HY_1(D, E(D)) = 0,$$

and the cokernel is $HY_0(D, E(D)) \cong E(D)/\langle u - v\rangle$. □

5.8 - Filtration on the cotangent complex. Given a dialgebra D, the length-filtration $F_k E(D)$ on the universal enveloping algebra $E(D)$, defined in (3.6), induces a filtration on the cotangent complex $EY_*(D)$, given by the family of subcomplexes $F_k EY_*(D)$, for $k \geq 0$, with chain modules

$$F_k EY_n(D) := k[Y_n] \otimes F_{k-n} E(D) \otimes D^{\otimes n} \qquad n \geq 0.$$

In fact the boundary operator on $EY_*(D)$ respects the filtration,

$$dF_k EY_n(D) \subset k[Y_{n-1}] \otimes F_{(k-n)+1} E(D) \otimes D^{\otimes n-1} = F_k EY_{n-1}(D),$$

because the faces d_0 and d_n increase the length of the elements of $E(D)$. These complexes have the standard properties $F_kEY_*(D) \subset F_{k+1}EY_*(D)$, for any $k \geq 0$, $\bigcup_{k\geq 0} F_kEY_*(D) = EY_*(D)$, and moreover $F_0EY_n(D) = k$, for $n = 0$, and $F_0EY_n(D) = 0$, for $n \geq 1$.

Let $GrEY_*(D) = \bigoplus_{k\geq 0} Gr^kEY_*(D)$ be the graded complex where $Gr^0EY_*(D) = F_0EY_*(D)$ and $Gr^kEY_*(D) = F_kEY_*(D)/F_{k-1}EY_*(D)$ for $k \geq 1$.

5.9 - Proposition. *Let V denote the underlying k-module of a dialgebra D. Regard V as an abelian dialgebra, with $a \star b := 0$ for any $a, b \in V$. Then there is an isomorphism of complexes*

$$GrEY_*(D) \cong EY_*(V).$$

Proof. First we describe the isomorphism of k-modules $GrEY_n(D) \cong EY_n(V)$, for any $n \geq 0$. By definition we have

$$\begin{aligned} GrEY_n(D) &= \bigoplus_{k\geq 0}[F_kEY_n(D)/F_{k-1}EY_n(D)] \cong \bigoplus_{k\geq 0} k[Y_n] \otimes [F_{k-n}E(D)/F_{k-1-n}E(D)] \otimes D^{\otimes n} \\ &\cong k[Y_n] \otimes [\bigoplus_{k\geq 0} Gr^{k-n}E(D)] \otimes D^{\otimes n} \cong k[Y_n] \otimes E(V) \otimes V^{\otimes n} \end{aligned}$$

because of (3.8) and the isomorphism of k-modules $D \cong V$.

Now look at the boundary operators defined on the two complexes. The faces d_i on $GrEY_n(D)$ yield zero for $i = 1, ..., n-1$. In fact any element $y \otimes e \otimes \underline{a} \in GrEY_n(D)$ belongs to some component $Gr^kEY_n(D)$, and in particular e is an element of $F_{k-n}E(D)$. Then for $i = 1, ..., n-1$ the element $d_i(y \otimes e \otimes \underline{a}) = d_i(y) \otimes e \otimes (a_1, ..., a_i \circ_i^y a_{i+1}, ..., a_n)$ belongs to $Gr^kEY_{n-1}(D) = F_kEY_{n-1}(D)/F_{k-1}EY_{n-1}(D)$. This means that e is now seen as an element in the quotient $F_{k-n+1}E(D)/F_{k-n}E(D)$ and thus it is zero. Finally, the faces d_0 and d_n on $GrEY_n(D)$ do not involve the dialgebra structure of D, but only the product in the algebra $GrE(D)$, which is the same as in $E(V)$. □

5.10 - Corollary. *Let D be a dialgebra and $E(D)$ its universal enveloping algebra. Then $HY_0(D, E(D)) \cong E(D)/\langle u - v\rangle$ and*

$$HY_n(D, E(D)) = 0, \qquad n \geq 1.$$

Proof. Consider the cotangent complex of D, $EY_n(D) = k[Y_n] \otimes E(D) \otimes D^{\otimes n}$ and its graded complex $GrEY_*(D) = \bigoplus_{k\geq 0} Gr^kEY_*(D)$ where $Gr^0EY_*(D) = F_0EY_*(D)$ and $Gr^kEY_*(D) = F_kEY_*(D)/F_{k-1}EY_*(D)$ for $k \geq 1$. Since $F_0EY_n(D) = k$, for $n = 0$, and it is zero for $n \geq 1$, we know that $H_n(F_0EY_*(D)) = 0$ for any $n \geq 1$. By proposition (5.9) we know that $GrEY_*(D) \cong EY_*(V)$, where V is the abelian dialgebra on the underlying k-module of D.

Now, by proposition (5.7) we have $H_n(GrEY_*(D)) = H_n(EY_*(V)) = 0$. From $H_n(F_0EY_*(D)) = 0$ and $H_n(F_1EY_*(D)/F_0EY_*(D)) = 0$ it follows that $H_n(F_1EY_*(D)) = 0$ for any $n \geq 1$. So on we proceed showing that $H_n(F_kEY_*(D)) = 0$ for any $k \geq 0$ and $n \geq 1$. Hence $H_n(\bigoplus_{k\geq 0} F_kEY_*(D)) = 0$ for any $n \geq 1$. □

6 Homology of bar-unital dialgebras

As we saw in (4.11), the bar homology of a bar-unital dialgebra is always zero. In this case, a bar-unit, for dialgebras, plays the same role as the unit for associative algebras. In this section we investigate this precise role, and its consequences on homology.

The dialgebra chain complex has boundary operator given by means of face maps. Hence the family of chain modules has an underlying structure of pre-simplicial module. For associative algebras, a unit permits us to define the degeneracy maps, and the Hochschild chain complex is in fact a simplicial module. For dialgebras, the bar-unit enables us to define similar maps. It turns out that the dialgebra chain modules are not simplicial, but the weaker properties of the degeneracy maps, called almost-simplicial, are enough to prove important theorems like Morita invariance of dialgebra homology for matrices and the isomorphism between dialgebra and Hochschild homology for unital associative algebras, proved in [F2].

6.1 - Almost-simplicial modules. An *almost-simplicial* module is a collection of k-modules $M = \{M_n\}_{n\geq 0}$ equipped with face maps $d_i : M_n \longrightarrow M_{n-1}$, for $i = 0, ..., n$, satisfying relations

$$(d) \qquad d_i d_j = d_{j-1} d_i, \qquad i < j,$$

and degeneracy maps $s_j : M_n \longrightarrow M_{n+1}$, for $j = 0, ..., n$, satisfying relations

$$(ds) \quad d_i s_j = \begin{cases} s_{j-1} d_i, & i < j, \\ id, & i = j, j+1, \\ s_j d_{i-1}, & i > j+1, \end{cases}$$

$$(s) \qquad s_i s_j = s_{j+1} s_i, \qquad i < j \qquad (\text{and not } i \leq j).$$

A simplicial module is an almost-simplicial module such that $s_i s_i = s_{i+1} s_i$, for any $i = 0, ..., n$.

Being pre-simplicial, an almost-simplicial module $M = \{M_n, d_i\}$ gives rise to a chain complex (M_*, d), with boundary operator $d = \sum_{i=0}^{n} (-1)^i d_i$. Being pseudo-simplicial[3], an almost-simplicial module satisfies the Eilenberg-Zilber Theorem described in [F2] and proved in [I].

6.2 - Degeneracies on dialgebra chain modules. Let D be a dialgebra with a bar-unit 1 and a corepresentation N. For any $n \geq 0$ and any $j = 0, ..., n$, define the degeneracy $s_j : CY_n(D, N) \longrightarrow CY_{n+1}(D, N)$ by

$$s_j(y \otimes (a_0, a_1, ..., a_n)) := s_j(y) \otimes (a_0, a_1, ..., a_j, 1, a_{j+1}, ..., a_n),$$

where a_0 is taken in N, $a_1, ... a_n$ in D, and for any $j = 0, ..., n$, the map $s_j : Y_n \longrightarrow Y_{n+1}$ *bifurcates* the j^{th} leaf of an n-tree y (that is, the tree $s_j(y)$ has the j^{th} leaf $|$ replaced with the branch $\curlyvee$).

6.3 - Lemma. *If the D-corepresentation N is unital, i.e.* $1 \Vdash n = n = n \dashV 1$ *for all* $n \in N$, *the degeneracy maps satisfy relations* (ds) *and* (s), *for* $i < j$. *Hence the chain complex* $CY_*(D, N)$ *of a bar-unital dialgebra has an underlying structure of almost-simplicial module.*

Proof. (ds) The operations $d_i s_j$ on a tree y first adds a leaf replacing the leaf number j with the branch $\curlyvee$, and then deletes the leaf number i. So, when $i < j$, we obtain the same tree if we first delete the i^{th} leaf and then bifurcate the original j^{th} leaf, which is now labeled by $j - 1$. Hence the two terms

$$d_i s_j(y \otimes \underline{a}) = d_i s_j(y) \otimes (a_0, ..., a_i \circ_i^{s_j(y)} a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n)$$

and

$$s_{j-1} d_i(y \otimes \underline{a}) = s_{j-1} d_i(y) \otimes (a_0, ..., a_i \circ_i^{y} a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n).$$

[3] By definition, a *pseudo-simplicial* module is a family of k-modules endowed with faces and degeneracies satisfying relations (d) and (ds) but not necessarily relations (s), see [T-V] and [I].

are equal because of the equality $a_i \circ_i^{s_j(y)} a_{i+1} = a_i \circ_i^{y} a_{i+1}$ for $i < j$. For $i = j$ we have:

$$d_i s_i(y \otimes \underline{a}) = d_i s_i(y) \otimes (a_0, ..., a_i \circ_i^{s_i(y)} 1, a_{i+1}, ..., a_n) = y \otimes \underline{a},$$

because the operator d_i evidently brings the tree $s_j(y)$ (with branch $\curlyvee$ labeled by $j, j+1$) back to the original tree and $\circ_i^{s_i(y)} = \dashv$. Similarly, for $i = j+1$ the identity $d_{j+1} s_j(y \otimes \underline{a}) = y \otimes \underline{a}$ follows from $\circ_{j+1}^{s_j(y)} = \vdash$. Finally, for $i > j+1$ the identity follows from the equality $a_{i-1} \circ_i^{s_j(y)} a_i = a_{i-1} \circ_i^{y} a_i$ for $i > j$, and the fact that we can invert the operations $d_i s_j$ and $s_{j-1} d_i$ after having observed that the leaf number i of the tree $s_j(y)$ is the leaf number $i-1$ in the tree y.

(s) Take $i < j$. The equality of the terms

$$s_i s_j(y \otimes \underline{a}) = s_i s_j(y) \otimes (a_0, ..., a_i, 1, a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n)$$

and

$$s_{j+1} s_j(y \otimes \underline{a}) = s_{j+1} s_j(y) \otimes (a_0, ..., a_i, 1, a_{i+1}, ..., a_j, 1, a_{j+1}, ..., a_n)$$

follows from the fact that the tree $s_i s_j(y)$, obtained from y bifurcating first the leaf number j and then the leaf number i, produces the same tree by performing the two bifurcations in inverted order. (Observe that the j^{th} leaf of y is the leaf number $j+1$ of $s_i(y)$).

Notice that for $i = j$ the operator $s_i s_i$ replace the i^{th} leaf with the branch $\curlyvee\!\!/$, while the operator $s_{i+1} s_i$ produces the branch $\backslash\!\!\curlyvee$, hence they can not coincide. $\square$

6.4 - Definition (normalized dialgebra homology). Given a bar-unital dialgebra D and a unital corepresentation N, we have the subcomplex of *degenerate elements* of $CY_*(D, N)$,

$$DY_n(D, N) := \{s_j(y, \underline{a}) \in CY_n(D, N) \mid \text{for any } (y, \underline{a}) \in CY_{n-1}(D, N) \text{ and any degeneracy } s_j\}.$$

The family of quotients $\bar{C}Y_n(D, N) := CY_n(D, N)/DY_n(D, N)$ is also a complex, that we call *normalized complex* of the dialgebra D. Its homology

$$\bar{H}Y_n(D, N) := H_n(\bar{C}Y_*(D, N), d)$$

is called *normalized homology of D*.

With no substantial differences from the case of associative algebras we have the following isomorphism.

6.5 - Proposition. *The complex of degenerate elements $DY_*(D, N)$ is acyclic, and the canonical projections $CY_n(D, N) \longrightarrow \bar{C}Y_n(D, N)$ induce an isomorphism on homology,*

$$HY_*(D, N) \cong \bar{H}Y_*(D, N).$$

The Eilenberg-Zilber Theorem for pseudo-simplicial modules, proved in [I], enabled us, in [F2], to prove the isomorphism between Hochschild and dialgebra homology for associative unital algebras,

$$HY_n(A, M) \cong H_n(A, M), \qquad n \geq 0.$$

From (5.1) it follows immediately that Hochschild homology of unital algebras can be interpreted as the derived functor of the tensor bifunctor $\otimes_{E(A)}$ on the category of modules over $E(A)$, and not only as derived functor of the tensor $\otimes_{A^e}$ on the category of modules over the universal enveloping algebra A^e.

6.6 - Corollary. *For any associative unital algebra A, and any unital A-bimodule M, we have isomorphisms*

$$H_n(A,M) \cong \mathrm{Tor}_n^{E(A)}(A,M) \qquad \textit{and} \qquad H^n(A,M) \cong \mathrm{Ext}^n_{E(A)}(A,M).$$

The Eilenberg-Zilber Theorem for pseudo-simplicial modules allows us to prove Morita invariance of dialgebra homology for matrices for any bar-unital dialgebra and *any* unital corepresentation, avoiding the restrictions needed in [R] to use the generalized trace map.

6.7 - Theorem (Morita Invariance). *Let D be a dialgebra and N a D-corepresentation. Consider, for any $r \geq 1$, the dialgebra of $r \times r$ matrices $M_r(D)$ with entries in D, and its corepresentation $M_r(N)$. If D and N are bar-unital, there is an isomorphism on dialgebra homology*

$$HY_n(M_r(D), M_r(N)) \cong HY_n(D,N), \qquad n \geq 0.$$

Proof. The proof is based on the isomorphism of k-vector spaces $\mathrm{M}_r(D) \cong \mathrm{M}_r(k) \otimes D$. Call (C^1_*, b^1) the Hochschild complex of the associative unital algebra $\mathrm{M}_r(k)$ with coefficients in itself, i.e. with the chain modules $C^1_n = \mathrm{M}_r(k)^{\otimes n+1}$ and the Hochschild boundary. We know that the complex C^1_* is associated to a simplicial module, that we denote by C^1, and that it is acyclic, i.e. $H_0(C^1_*) = k$ and $H_n(C^1_*) = 0$ for any $n \geq 1$.

Call (C^2_*, b^2) the dialgebra complex of D with coefficients in the corepresentation N, i.e. with the chain modules $C^2_n = k[Y_n] \otimes N \otimes D^{\otimes n}$ and the dialgebra boundary. When D and N are bar-unital, the complex C^2_* is associated to an almost-simplicial module that we denote by C^2. Then we can apply the Eilenberg-Zilber Theorem, [F2], and conclude that

$$H_n(\Delta(C^1_* \otimes C^2_*), b^1 \otimes b^2) = H_n(C^1 \times C^2) \cong H_n(C^2_*, b^2) = HY_n(D,N)$$

for any $n \geq 0$, where $\Delta(C^1_* \otimes C^2_*)$ is the diagonal complex of the tensor product of the two complexes.

Finally the result follows from the fact that the dialgebra complex of $\mathrm{M}_r(D)$ with coefficients in $\mathrm{M}_r(N)$ is isomorphic to the diagonal complex $(\Delta(C^1_* \otimes C^2_*), b^1 \otimes b^2)$. □

Remark that the same trick can be used to prove Morita Invariance for Hochschild homology.

6.8 - Theorem (Künneth formula). *Let D be a dialgebra with a corepresentation N, and let A be an associative algebra with a bimodule M. If all these spaces are (bar-)unital, we have an isomorphism on homology*

$$HY_*(A \otimes D, M \otimes N) \cong HY_*(A,M) \otimes HY_*(D,N).$$

Proof. As usual, the module of n-chains of the dialgebra $A \otimes D$,

$$CY_*(A \otimes D, M \otimes N) = k[Y_n] \otimes M \otimes N \otimes A^{\otimes n} \otimes D^{\otimes n} = (M \otimes A^{\otimes n}) \otimes (k[Y_n] \otimes N \otimes D^{\otimes n}),$$

splits into the product of two ps-modules: one is Hochschild chain module of A, and the other is the dialgebra chain module of D. By the Eilenberg-Zilber Theorem for pseudo-simplicial modules, we then have an isomorphism

$$HY_*(A \otimes D, M \otimes N) \cong H_*(A,M) \otimes HY_*(D,N),$$

where $H_*(A,M) \cong HY_*(A,M)$. □

7 HY-unital dialgebras

An associative algebra without unit is a dialgebra without bar-units, and gives rise to a chain complex which is not an almost-simplicial module. Hence, to handle its dialgebra homology, we can only make use of the spectral sequence. Even for associative algebras, the total homology of the vertical towers described in (5.5) is not easy to compute, and in general there is no relationship with the Hochschild homology of the algebra itself. The typical example is provided by the free non-unital associative algebra.

7.1 - Example (free associative algebra). Let $Ass(V) = \bar{T}(V) = V \oplus V^{\otimes 2} \oplus \cdots$ be the free associative non-unital algebra on a k-module V. It is well known that Hochschild homology vanishes on the free algebra from the second-order group, that is $HH_n(Ass(V)) = 0$, for $n \geq 2$. Instead, already in small dimension, the dialgebra homology *does not vanish* on the free associative algebra, and we find

$$\begin{aligned}
&HHY_0(\bar{T}(V)) \cong HH_0(\bar{T}(V)) = \textstyle\bigoplus_{n\geq 1} V^{\otimes n}/(id-\tau) \quad \text{(coinvariants)},\\
&HHY_1(\bar{T}(V)) \cong HH_1(\bar{T}(V)) = \textstyle\bigoplus_{n\geq 2} (V^{\otimes n})^{\tau} \quad \text{(invariants)},\\
&HHY_2(\bar{T}(V)) = V^{\otimes 3},\\
&HHY_3(\bar{T}(V)) = 2V^{\otimes 4}.
\end{aligned}$$

A class of algebras on which the computations are particularly simple, since almost all terms of the E^1-plane are already zero, is that of homologically unital algebras. For associative algebras, the property of having a unit is stronger then the property of having acyclic bar complex. An algebra A such that $H_*(A) = 0$, without necessarily having a unit, is called *homologically unital*, or *H-unital*. The importance of H-unital algebras is related to the extensions of unital algebras, since the kernel of such extensions can not be a unital algebra. It was proved by Wodzicki ([W], [L1]), that H-unitarity is equivalent to the excision property for Hochschild homology.

In [F2] we proved that for associative unital algebras there is an isomorphism between dialgebra and Hochschild homology with coefficients in unitary bimodules. We can now extend this result to the class of H-unital algebras.

7.2 - Theorem. *Let A be an associative H-unital algebra and M an A-bimodule. The dialgebra homology of A with values in M is isomorphic to the Hochschild homology of A, that is*

$$HY_n(A, M) \cong H_n(A, M), \qquad n \geq 0.$$

Proof. Consider the bicomplex $CY_{p,q}(A, M) = k[Y_{p,q}] \otimes M \otimes A^{\otimes p+q+1}$ and the spectral sequence converging to the dialgebra homology of A.

1) Let us compute the E^1-plane, $E^1_{pq} = \bigoplus_{y \in Y_p} H_q(CY^{\tilde{y}}_{p,*}(A, M), d^v)$.

• For $p = 0$, the vertical complex $CY_{0,q}(A, M)$ is isomorphic to the Hochschild complex of A shifted by 1. Hence

$$\begin{aligned}
&E^1_{00} = \mathrm{coker}(M \otimes A^{\otimes 2} \longrightarrow M \otimes A),\\
&E^1_{0q} \cong H_{q+1}(A, M), \qquad q \geq 1.
\end{aligned}$$

• For $p \geq 1$ remark that in the multi-complex $CY_{p,*}(A, M)$ any vertical boundary operator contains either the first face map, d_0, or the last one, d_{p+1+*}, or none of the two. So the first and the last face maps never appear together. This means that the homology along each direction of the vertical tower is in fact a bar-homology for A, so by assumption it is zero. Hence the total homology of each vertical tower is zero, and finally

$$E^1_{p0} \cong \bar{M} \otimes_A A \otimes_A \cdots \otimes_A A,$$
$$E^1_{pq} = 0, \qquad q \geq 1,$$

where $\bar{M} = M/AM$ is the module of coinvariants of M by the left action of A.

2) The only non-zero horizontal boundary which remains is

$$\cdots \longrightarrow E^1_{p0} \longrightarrow E^1_{p-10} \longrightarrow \cdots \longrightarrow E^1_{p0},$$

where $E^1_{p0} \cong \bar{M} \otimes_A A \otimes_A \cdots \otimes_A A$. The homology of this complex yields

$$\tilde{M} \otimes A/A^2 \otimes \cdots A/A^2,$$

where $\tilde{M} = \bar{M}/MA$ is the module of coinvariants of M by both left and right actions of A. This homology is zero since the algebra A is supposed to be H-unital. In conclusion the only non-zero terms of the E^2-plane are given by the vertical complex at $p = 0$, which is the Hochschild complex of A shifted by 1.

3) Finally, taking into account the homology of the external map,

$$\cdots \longrightarrow E^1_{00} \longrightarrow M,$$

we obtain the isomorphism $HY_n(A, M) \cong H_n(A, M)$ for any $n \geq 0$. □

The notion of homologically unital algebras can naturally be extended to the category of dialgebras.

7.3 - HY-unital dialgebras. A dialgebra D is called *HY-unital* if $HY_n(D) = 0$ for all $n \geq 1$. Theorem (4.11) implies that bar-unital dialgebras are HY-unital.

A standard example of an HY-unital dialgebra which has no bar-unit is provided by the inductive limit $\mathrm{M}(D) = \mathrm{M}_{r\to\infty}(D)$ of the inclusions of matrix dialgebras $\mathrm{M}_r(D) \hookrightarrow \mathrm{M}_{r+1}(D)$ with entries in a bar-unital dialgebra. The identity $HY_*(\mathrm{M}(D)) = 0$ is a consequence of Morita invariance of dialgebra homology with constant coefficients, namely $HY_*(\mathrm{M}(D)) \cong HY_*(D)$, which can be proved as we did for theorem (6.7).

From (7.2) it follows immediately that an associative algebra which is H-unital is also HY-unital. The converse is also true.

7.4 - Theorem. *An associative algebra A is H-unital if and only if A is HY-unital, that is*

$$H_*(A) = 0 \quad \Longleftrightarrow \quad HY_*(A) = 0.$$

Proof. Assume that $HY_n(A) = 0$ for any $n \geq 1$. Then $H_0(A, k) \cong HY_0(A, k) = k$, where k is the trivial bimodule over A, and $H_1(A) \cong HY_1(A)$, which is zero by assumption. We prove that $H_n(A) = 0$ recursively on the order n.

Consider the bicomplex labeled by the oriented trees. We know on one side that $HY_n(A)$ is the total homology of this bicomplex, and on the other side that each vertical complex has homology given by means of Hochschild homology groups at smaller orders.

In particular, $HY_2(A)$ is the direct sum of $H_2(A)$, in the position $(0,1)$ of the bicomplex, and some additional terms involving only $H_1(A)$ placed in the position $(1,0)$. Since $HY_2(A)$ is zero by assumption, and $H_1(A) \cong HY_1(A) = 0$, we get also $H_2(A) = 0$.

Similarly, $HY_3(A)$ is the direct sum of $H_3(A)$, in the position $(0,2)$ of the bicomplex, and some additional terms involving only $H_1(A)$ and $H_2(A)$, placed in the position $(1,1)$ and $(2,0)$. Hence $H_3(A) = 0$. In this way, proceding step by step, one gets $H_n(A) = 0$ for all $n > 0$. $\square$

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