Trees and renormalization of QED

ALESSANDRA FRABETTI

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Abstract

The Green functions of the electron and photon propagators of QED satisfy a system of Schwinger functional equations which can be solved perturbatively with a tree expansion method, as described by Ch. Brouder.

This method is compatible with a map which can be given between trees and Feynman diagramms, and leads to study the renormalization procedure of the QED propagators directly on trees. After renormalization, the coefficients of the tree expansion can still be described recursively, by means of a pruning coproduct operator defined on the set of trees.

The comparision between the coefficients before and after renormalization is again possible by virtue of a Hopf algebra defined on trees. This gives a noncommutative version of the Hopf algebra of renormalization found by A. Connes and D. Kreimer for scalar field theories.

1 Tree expansion for Schwinger equations

Introduction

Quantum electrodynamics (QED) is the quantization of the classical electrodynamic fields.

Roughly speaking, the dynamics of a classical field theory $\phi = \phi(x)$ on a basic manifold with coordinates x (typically the Minkowski flat space) is described by a Lagrangian density $\mathcal{L}(\phi)$ which contains the informations of the kinetic energy (for the free field) and the potential energy (for the interactions). The Lagrangian determines an action density $\mathcal{S}(\phi;\gamma) = \int_\gamma \mathcal{L}(\phi)dx$, which depends on the domain γ of integration, for which only the initial and final points are fixed. The equations of motion for ϕ are differential equations obtained by applying the principle of least action $d\mathcal{S}(\phi;\gamma) = 0$, which says that the effective trajectory covered by the field is the one which minimizes the action. The resulting equations

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

are called Euler-Lagrange equations.

The quantization of the field ϕ follows the standard prescription of quantization: the field itself has to be regarded as an operator valued distribution and the classical observables as functions of this field and its derivatives. In particular, the multiplication by ϕ corresponds to the creation (annihilation) of the particle state corresponding to the field, $|x >= \phi(x)|0 >$. (In fact, if $\dot\phi(x) = \frac{\partial \phi(x)}{\partial t}$, the operator $\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot\phi}$ is the momentum density conjugated to ϕ, and satisfies the uncertainty principle identity $[\phi(x), \Pi(y)] = i\delta(x-y)$.)

The field itseld can not be measured. What can be measured is the probability that a state x propagates to a state y. This quantity is related to the 2-points correlation function $G(x,y)$, which is the Green function of the differential operator applied to the field ϕ in the Euler-Lagrange equation. Its precise value is known only for free Lagrangians, i.e. when the interaction potential is zero.

When the interaction potential is non zero but it is very small compared to the kinetic energy, we can treat it as a small perturbation of the free case. Then, the Green function $G(x,y)$ becomes a power series $G(x,y) = \sum_{n=0}^{\infty} g^n G_n(x,y)$ on the small real parameter g, the "coupling constant" which characterizes the interaction.

The usual way to determine the coefficients $G_n(x,y)$ of this expansion is due to Feynman: $G_n(x,y)$ is a finite sum of integrals completely described by all Feynman diagrams with n loops admitted by the specified potential energy of the field theory. A list of Feynman's rules and the types of vertices occurring in the Feynman graphs for some standard field theories can be found in any textbook of quantum field theory, for instance [IZ].

For QED, Ch. Brouder proposed a perturbative method of resolution based on planar binary trees instead of Feynman's graphs.

Schwinger equations

The Green functions $S(q)$ and $D_{\mu\nu}(q)$ of the electron and photon propagators (after a Fourier transformation) are 4×4 complex matrices which satisfy a system of functional equations (Schwinger equations)

$$\left\{ \begin{array}{rl} S(q) = & S^0(q) + ie_0 S^0(q) \displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \gamma^\lambda D_{\lambda\lambda'}(p) \frac{\delta S(q-p)}{\delta A^0_{\lambda'}(p)} \\ D_{\mu\nu}(q) = & D^0_{\mu\nu}(q) - ie_0 D^0_{\mu\lambda}(q) \displaystyle\int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{tr}\big[\gamma^\lambda \frac{\delta S(p)}{\delta A^0_{\lambda'}(-q)}\big] D_{\lambda'\nu}(q) \end{array} \right. \tag{1}$$

with initial conditions given by the free fields

$$\left\{ \begin{array}{rl} S^0(q) = & (\gamma^\mu q_\mu - m + i\epsilon)^{-1} \\ D^0_{\mu\nu}(q) = & -\frac{g_{\mu\nu}}{q^2+i\epsilon} + (1 - 1/\xi) \frac{q_\mu q_\nu}{(q^2+i\epsilon)^2} \end{array} \right.$$

A Schwinger equation is a functional equation where the unknown $X = X(A)$ is a functional on the functions (or distributions) $A(p)$ of the coordinates p. The functional X may depend on some external parameters q and may have values in a field of scalars or in an associative algebra. A typical Schwinger equation is of the form

$$X = X_0 + F_p(X, \frac{\delta X}{\delta A(p)}) \tag{2}$$

where X_0 is a given functional which represents the initial conditions, F_p is a linear operator in each of its variable, which typically integrates over the coordinate p, and $\frac{\delta X}{\delta A(p)}$ is the functional derivative of X with respect to A in the direction of the Dirac delta distribution $\delta_p(x) = \delta(x-p)$, that is

$$\frac{\delta X}{\delta A(p)} = \lim_{\epsilon \to 0} \frac{X(A + \epsilon\delta_p) - X(A)}{\epsilon}.$$

The equations of QED are in fact a system of Schwinger equations of the form

$$\begin{cases} X &= X_0 + X_0 F_p(Y, \dfrac{\delta X}{\delta A(p)}) \\ Y &= Y_0 + Y_0 G_p(Y, \dfrac{\delta X}{\delta A(p)}) \end{cases} \tag{3}$$

with initial conditions X_0 and Y_0.

To solve equation (2), and similarly for the system (3), we need the successive functional derivatives of X. Hence a Schwinger equation hides an infinite system of functional equations

$$\begin{aligned} X = & \quad X_0 + F_p(X, \frac{\delta X}{\delta A(p)}) \\ \frac{\delta X}{\delta A(p_1)} = & \quad \frac{\delta A}{\delta v(p_1)} + F_p(\frac{\delta X}{\delta A(p_1)}, \frac{\delta X}{\delta A(p)}) + F_p(X, \frac{\delta^2 X}{\delta A(p_1)\delta A(p)}) \\ & \dots \end{aligned}$$

with initial values

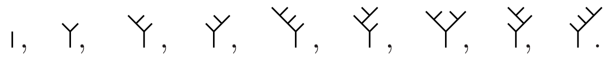
$$X_0, \quad \frac{\delta X_0}{\delta A(p_1)}, \quad \frac{\delta^2 X_0}{\delta A(p_1)\delta A(p_2)}, \quad \dots$$

Such equations are solved with perturbative methods. The first method was used in quantum field theory for more the 50 years now, and consists of an expansion $X = \sum_\Gamma g^{|\Gamma|} x(\Gamma)$ as power series of the coupling constant g ($g = e_0^2$ for QED), labelled by Feynman graphs Γ. The computation of the coefficients $x(\Gamma)$ becomes soon very complicated already for graphs with few loops.

A second method was proposed by Christian Brouder in [B], it is based on the expansion over planar binary trees. Its main advantage lays in the fact of producing explicit recursive solutions for the coefficients of the expansion.

Tree-expanded solutions

Planar binary trees are planar connected graphs without cycles, with trivalent internal vertices and a preferred external vertex called the root. The set of all such trees is denoted by Y, cf. [F], [LR]. Such trees t are naturally graded by the number $|t|$ of internal vertices (or equivalently by the number of the external ones, called the leaves). Here are all the planar binary trees, in the sequel called "trees", up to degree 3:



Each tree t, except for the "root tree" $|$, determines uniquely two new trees t_l and t_r of smaller order such that t is the grafting $t_l \vee t_r$ on a new root.

The perturbative solution of equation (2) is as follows.

Theorem 1.1 [B] *Let $x(t)$ be the sequence of functionals of $A(p)$ indexed by $t \in Y$ and recursively defined by $x(|) = X_0$ and*

$$x(t) = F_p(x(t_l), \frac{\delta x(t_r)}{\delta A(p)}),$$

for any tree $t \neq |$.

Then the functional $X = \sum_{t \in Y} e^{|t|} x(t)$, where e is any given constant, is a solution of the Schwinger equation (2).

The explicit solutions of equations (1) for the tree expansion of QED propagators is the following. Consider the tree expansion

$$\begin{cases} S(q) = & \sum_t e_0^{2|t|} \varphi(t;q) \\ D_{\mu\nu}(q) = & \sum_t e_0^{2|t|} \varphi_{\mu\nu}(t;q) \end{cases}$$

then the recursive solution for the coefficients is

$$\begin{cases} \varphi(t;q) = & i\varphi(|;q) \displaystyle\int \frac{\mathrm{d}^4p}{(2\pi)^4} \gamma^\lambda \varphi_{\lambda\lambda'}(t_l;p) \frac{\delta\varphi(t_r;q-p)}{e_0 \delta A^0_{\lambda'}(p)} \\ & \\ \varphi_{\mu\nu}(t;q) = & -i\varphi_{\mu\lambda}(|;q) \displaystyle\int \frac{\mathrm{d}^4p}{(2\pi)^4} \mathrm{tr}\big[\gamma^\lambda \frac{\delta\varphi(t_r;p)}{e_0 \delta A^0_{\lambda'}(-q)}\big] \varphi_{\lambda'\nu}(t_l;q), \end{cases}$$

with initial data given by the free propagators

$$\begin{cases} \varphi(|;q) = & S^0(q) & \dfrac{\delta\varphi(|;q)}{e_0\delta A^0_\lambda(p)} = & S^0(q)\gamma^\lambda S^0(q+p) \\ & & & \\ \varphi_{\mu\nu}(|;q) = & D^0_{\mu\nu}(q) & \dfrac{\delta\varphi_{\mu\nu}(|;q)}{e_0\delta A^0_\lambda(p)} = & 0. \end{cases}$$

Feynman diagrams and planar binary trees

The basic interpretation for QED is as follows. Each planar binary tree describes a propagator of photons or electrons and corresponds to a finite sum of Feynmann graphs which can not be distinguished by experiment.

Formally, this amount to give to linear maps

$$\sigma^\gamma : \mathbb{C}Y \longrightarrow \mathbb{C}\mathcal{F}^\gamma, \qquad \sigma^e : \mathbb{C}Y \longrightarrow \mathbb{C}\mathcal{F}^e,$$

which are sections (up to an integer factor) of two projections

$$\pi^\gamma : \mathbb{C}\mathcal{F}^\gamma \longrightarrow \mathbb{C}Y, \qquad \pi^e : \mathbb{C}\mathcal{F}^e \longrightarrow \mathbb{C}Y,$$

form the sets of 2-legs Feynman diagrams of QED to the set of trees.

Intuitively, the rule to construct the tree $\pi(\Gamma)$ associated to a given diagram is the following:

- The free propagator on which the perturbative terms are inserted corresponds to the root of the tree.
- We read the graph from the left to the right, and we add a leaf each time we meet a loop.
- The propagator of the loop determines the orientation of the leaf to graft: a photon loop corresponds to a \-leaf, an electron loop corresponds to a /-leaf.

Examples at small order are in the next figures.

Figure 1: Small perturbative terms for the electron Green function.

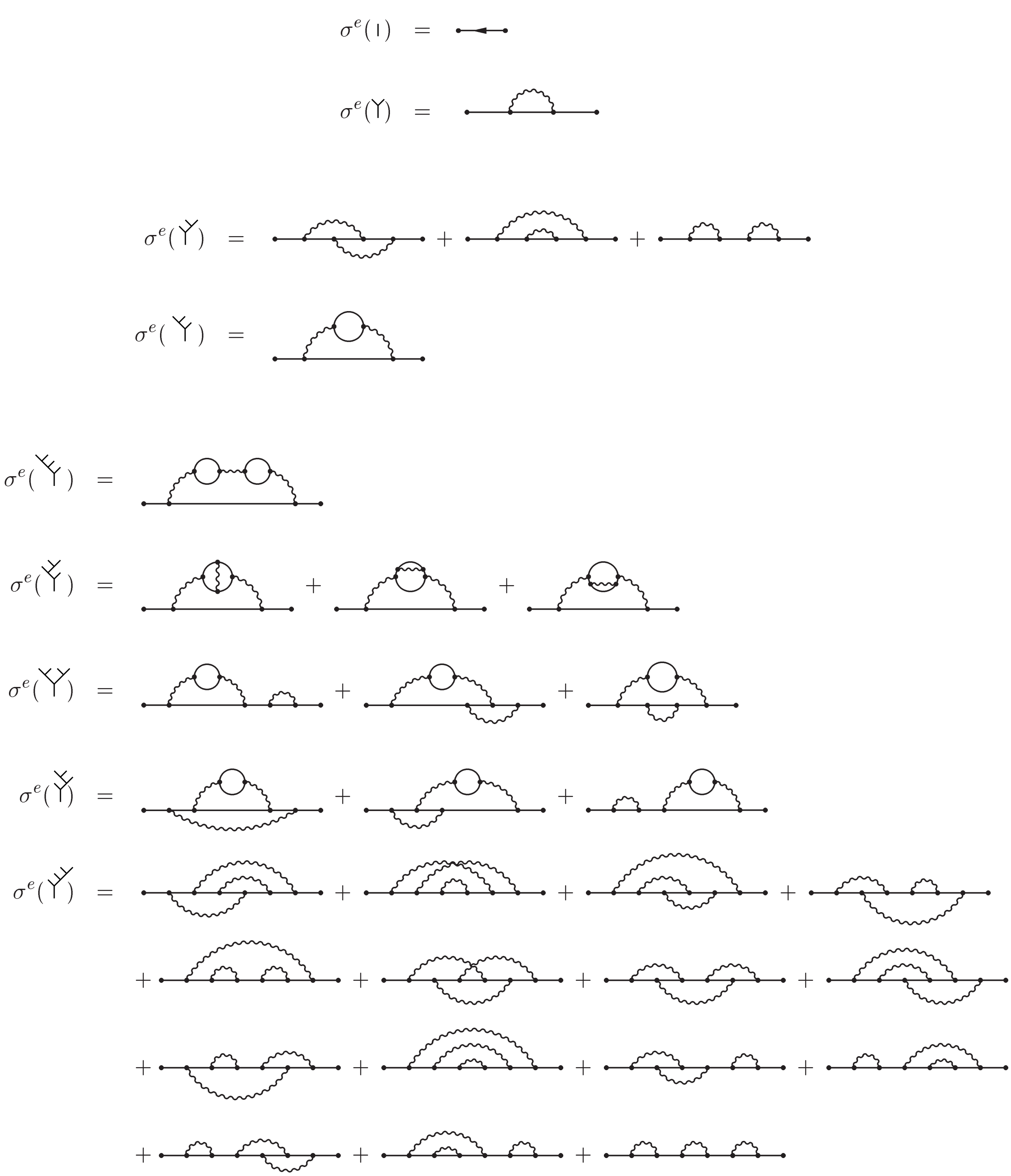
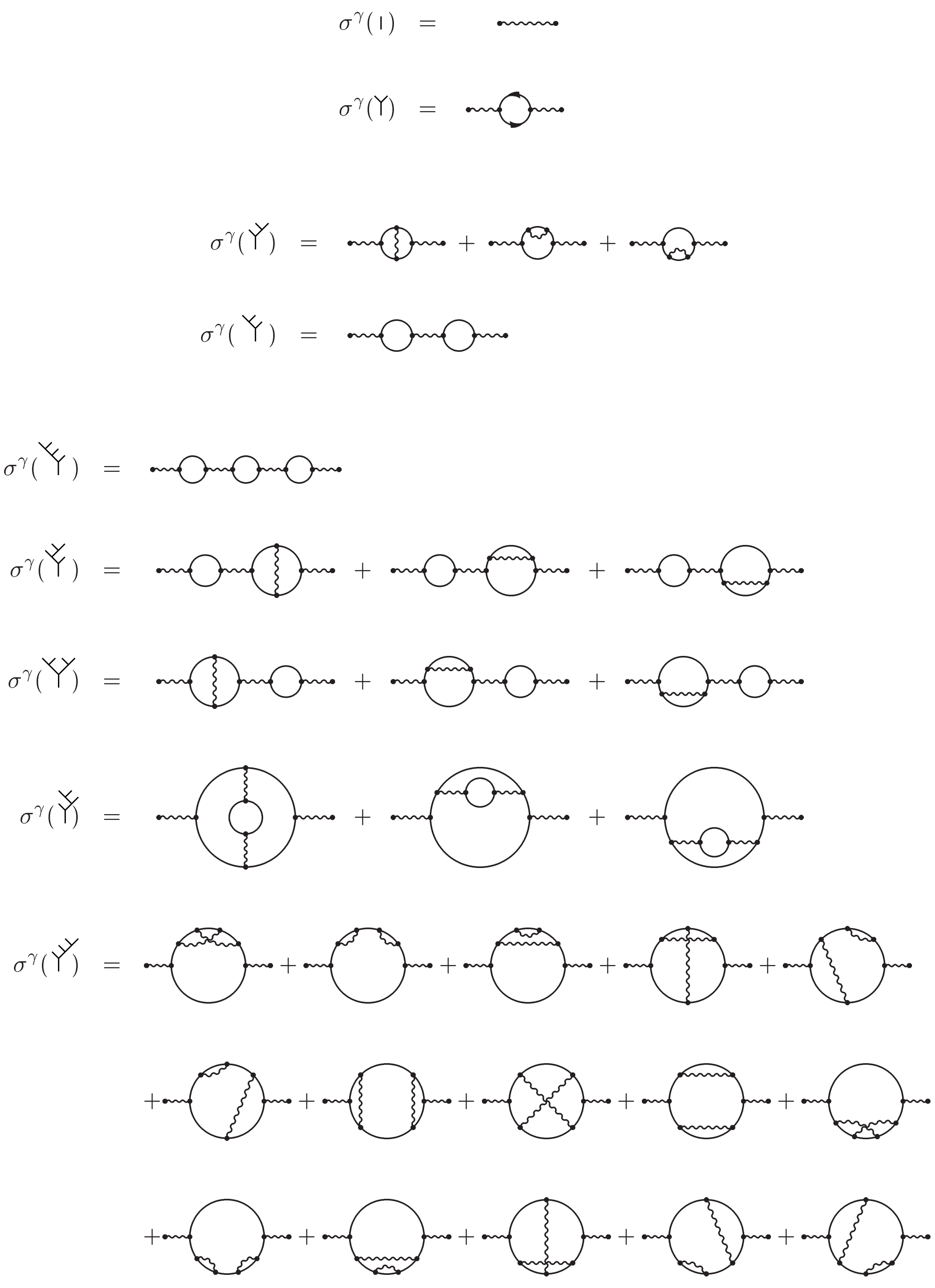


Figure 2: Small perturbative terms for the photon Green function.



The problem of renormalization

The terms described by 2-legs Feynmann graphs with at least one loop are all divergent, and need to be renormalized. The process of renormalization adds some interaction terms to the original Lagrangian, which would otherwise remain unknown. Hence it must be considered as a non negligible part of the quantization procedure.

For instance, the Lagrangian density which describes the interaction of the electron-positron field ψ, $\hat{\psi}$ with the quantized electromagnetic field A_0^μ has the form:

$$\mathcal{L}(\psi, A_0^\mu) = \quad -\tfrac{1}{4}F_0^{\mu\nu}F_{0\mu\nu} - \tfrac{\xi}{2}(\partial_\mu A_0^\mu)^2 + \hat{\psi}(i\gamma_\mu\partial^\mu - m_0)\psi - e_0\hat{\psi}\gamma_\mu A_0^\mu\psi,$$

where $F_0^{\mu\nu} = \partial^\mu A_0^\nu - \partial^\nu A_0^\mu$. After renormalization, in terms of the renormalized fields $\bar{\psi}$, $\bar{\hat{\psi}}$ and A^μ, it becomes:

$$\begin{aligned}\mathcal{L}(\bar{\psi}, A^\mu) = \quad & -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \tfrac{\xi}{2}(\partial_\mu A^\mu)^2 + \bar{\hat{\psi}}(i\gamma_\mu\partial^\mu - m)\bar{\psi} - e\bar{\hat{\psi}}\gamma_\mu A^\mu\bar{\psi} \\ & -(Z_3-1)\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + (Z_2-1)\bar{\hat{\psi}}(i\gamma_\mu\partial^\mu - m)\bar{\psi} - e(Z_2-1)\bar{\hat{\psi}}\gamma_\mu A^\mu\bar{\psi} + Z_2\bar{\hat{\psi}}\delta m\bar{\psi},\end{aligned}$$

with $\delta m = m - m_0$.

The renormalization of the term described by each Feynmann graph is nowadays clear: on simple divergences it is given by various physical schemes (MS, dimensional, regularization, mass), on combined divergences it is given by the Forest Formula and its Hopf algebra version found by Kreimer and Connes, cf. [K, CK].

It is therefore natural to ask the following question: is there a procedure which describes at once the renormalization of all Feynmann graphs described by a given planar binary tree?

The answer is YES, as explained in next section.

2 Tree-expansion for renormalized Schwinger equations

Renormalized Schwinger equations

In terms of functional equations, the Green functions for the renormalized fields in QED satisfy a system of Schwinger equations of new type:

$$\left\{\begin{aligned}\bar{S}(q) = \quad & S^0(q) - (Z_2-1)\bar{S}(q) - \delta m Z_2 S^0(q)\bar{S}(q) \\ & + ieZ_2 S^0(q)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\gamma^\lambda \bar{D}_{\lambda\lambda'}(p)\frac{\delta\bar{S}(q-p)}{\delta A_{\lambda'}(p)} \\ \\ \bar{D}_{\mu\nu}(q) = \quad & D^0_{\mu\nu}(q) - (Z_3-1)(\bar{D}_{\mu\nu}(q) + \frac{1}{\xi}\frac{q_\mu q_\nu}{(q^2+i\epsilon)^2}) \\ & -ieZ_2 D^0_{\mu\lambda}(q)\int \frac{\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\big[\gamma^\lambda\frac{\delta\bar{S}(p)}{\delta A_{\lambda'}(-q)}\big]\bar{D}_{\lambda'\nu}(q)\end{aligned}\right. \tag{4}$$

which involve the renormalized propagators and the Dyson renormalization factors:

$$\begin{aligned} Z_2 \bar{S}(q) =& \quad S(q) \\ Z_3 \bar{D}_{\mu\nu}(q) =& \quad D_{\mu\nu}(q) + (1 - Z_3)\frac{1}{\xi}\frac{q_\mu q_\nu}{(q^2 + i\epsilon)^2} \\ Z_3 e_0^2 =& \quad e^2 \\ m_0 =& \quad m - \delta m \end{aligned}$$

These equations can be described by the prototype

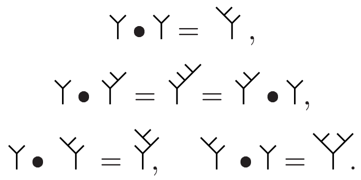
$$\bar{X}Z = X_0 + F_p(\bar{X}, \frac{\delta \bar{X} Z}{\delta A(p)}) + \delta m X_0 \bar{X} Z \tag{5}$$

where Z and δm are infinite constants containing a finite number of terms at each order of interaction.

The pruning coproduct

The perturbative solution of equation (5) can then be given with the help of a coproduct operator on the set of trees, cf. [BF1].

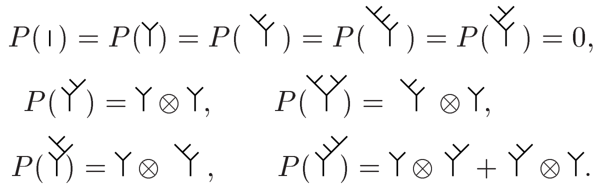
Let $\bullet : Y \times Y \longrightarrow Y$ be the associative product defined on each couple of trees (t, s) as the grafting $t \bullet s$ of the tree s on the rightmost leaf of t. Recursively: $| \bullet s := s$ and $t \bullet s := t_l \vee (t_r \bullet s)$. (Remark that the operation is called "under", $t \backslash s$, in [LR].) Examples of products are



Let $\mathbb{C}Y$ be the complex vector space on Y, and let $\Delta^P : \mathbb{C}Y \longrightarrow \mathbb{C}Y \otimes \mathbb{C}Y$ be the dual coproduct of $\bullet$. Then $\Delta^P(|) = | \otimes |$ and $\Delta^P(t) = t \otimes | + | \otimes t + P(t)$, where the pruning operator is recursively defined by $P(|) = 0$ and

$$P(t) = (t_l \vee |) \otimes t_r + \sum_{P(t_r)} (t_l \vee t_r^{(1)}) \otimes t_r^{(2)}$$

for $t \neq |$. The term $\Delta^P(t)$ can also be described as the sum over all possible cuts of /-oriented leaves in the right branch t_r. The cutted branch is then set on the right of the tensor product. The pruning operator is decribed in a similar way, just admitting only nontrivial cuts. For instance:



Lemma 2.1 *Let $T(Y)$ denote the tensor algebra over Y, modulo the identification of the unit 1 with the root tree $|$. If we extend multiplicatively Δ^P to tensor powers of trees, we obtain a Hopf algebra structure on $T(Y)$ with counit $\epsilon(|) = 1$ and $\epsilon(t) = 0$ for all $t \neq |$, and antipode determined by the recursive formula $S^P(t) = -t - \sum_{P(t)} S^P(t_{(1)} t_{(2)})$.*

Tree-expanded solutions

Let $\star$ denote the convolution associated to the coproduct Δ^P.

Theorem 2.2 [BF1] *Let $\bar{x}(t)$ be the sequence of functionals indexed by $t \in Y$ and recursively defined by $\bar{x}(\shortmid) = x(\shortmid) = X_0$ and from the formula*

$$(\bar{x} \star \zeta)(t) = F_p(\bar{x}(t_l), (\frac{\delta \bar{x}}{\delta A(p)} \star \zeta)(t_r)) + x(\shortmid)(\zeta_m \star \bar{x} \star \zeta)(t),$$

where we assume that the renormalization constants are expanded as $Z = \sum_{t \in Y} e^{|t|} \zeta(t)$ and $\delta m = \sum_{t \in Y} e^{|t|} \zeta_m(t)$, with initial values $\zeta(\shortmid) = 1$ and $\zeta_m(\shortmid) = 0$.

Then $\bar{X} = \sum_{t \in Y} e^{|t|} \bar{x}(t)$ is a solution of the renormalized Schwinger equation (5).

The explicit solutions of equations (4) for the tree expansion of QED propagators is the following. Consider the tree expansion

$$\begin{cases} \bar{S}(q) = & \sum_t e^{2|t|} \bar{\varphi}(t;q) \\ \bar{D}_{\lambda\mu}(q) = & \sum_t e^{2|t|} \bar{\varphi}_{\lambda\mu}(t;q) - \frac{1}{\xi} \frac{q_\mu q_\nu}{(q^2 + i\epsilon)^2} \\ Z_2 = & \sum_t e^{2|t|} \zeta_2(t), \qquad \zeta_2(\shortmid) = 1 \\ Z_3 = & 1 - \sum_{|t|>0} e^{2|t|} \zeta_3(t), \\ \delta m = & \sum_t e^{2|t|} \zeta_m(t), \qquad \zeta_m(\shortmid) = 0. \end{cases}$$

Then the recursive solution for the renormalized coefficients is

$$\begin{cases} \bar{\varphi}(t;q) = & -\zeta_2(t)\varphi(\shortmid;q) - \varphi(\shortmid;q)(\zeta_m \star \bar{\varphi}(q))(t) \\ & \qquad + i\varphi(\shortmid;q) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \gamma^\lambda \bar{\varphi}_{\lambda\lambda'}(t_l;p) \frac{\delta \bar{\varphi}(t_r;q-p)}{e\delta A_{\lambda'}(p)} \\ & \\ \bar{\varphi}_{\mu\nu}(\shortmid \vee t;q) = & \zeta_3(\shortmid \vee t)\varphi^T_{\mu\nu}(\shortmid;q) - i\varphi^T_{\mu\lambda}(\shortmid;q) \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \mathrm{tr}\big[\gamma^\lambda (\frac{\delta \bar{\varphi}(p)}{e\delta A_{\lambda'}(-q)} \star \zeta_2)(t)\big] \varphi^T_{\lambda'\nu}(\shortmid;q) \\ \bar{\varphi}_{\mu\nu}(t_l \vee t_r;q) = & \bar{\varphi}_{\mu\nu}(\shortmid \vee t_r;q)\bar{\varphi}_{\mu\nu}(t_l;q). \end{cases}$$

3 Hopf algebra for massless renormalization

Now we know that renormalization is possible on trees, and we know the coefficients before and after renormalization: what is the relation between them?

Here is a list of the explicit relations for $\delta m = 0$ up to order 3. For the electron Green function:

$$\bar\varphi(\Upsilon) = \varphi(\Upsilon) - \zeta_2(\Upsilon)\varphi(|)$$

$$\begin{aligned}
\bar\varphi(T_{2a}) &= \varphi(T_{2a}) + \zeta_3(\Upsilon)\varphi(\Upsilon) - \zeta_2(T_{2a})\varphi(|)\\
\bar\varphi(T_{2b}) &= \varphi(T_{2b}) - \zeta_2(\Upsilon)\varphi(\Upsilon) - \zeta_2(T_{2b})\varphi(|) + \zeta_2(\Upsilon)^2\varphi(|)
\end{aligned}$$

$$\begin{aligned}
\bar\varphi(T_{3a}) &= \varphi(T_{3a}) + 2\zeta_3(\Upsilon)\varphi(T_{2a}) + \zeta_3(\Upsilon)^2\varphi(\Upsilon) - \zeta_2(T_{3a})\varphi(|)\\
\bar\varphi(T_{3b}) &= \varphi(T_{3b}) + \zeta_3(T_{2b})\varphi(\Upsilon) - \zeta_2(T_{3b})\varphi(|)\\
\bar\varphi(T_{3c}) &= \varphi(T_{3c}) + \zeta_3(\Upsilon)\varphi(T_{2b}) - \zeta_2(\Upsilon)\varphi(T_{2a}) - \zeta_2(\Upsilon)\zeta_3(\Upsilon)\varphi(\Upsilon) + [-\zeta_2(T_{3c}) + \zeta_2(T_{2a})\zeta_2(\Upsilon)]\varphi(|)\\
\bar\varphi(T_{3d}) &= \varphi(T_{3d}) + \zeta_3(\Upsilon)\varphi(T_{2b}) - \zeta_2(T_{2a})\varphi(\Upsilon) + [-\zeta_2(T_{3d}) + \zeta_2(T_{2a})\zeta_2(\Upsilon)]\varphi(|)\\
\bar\varphi(T_{3e}) &= \varphi(T_{3e}) - \zeta_2(\Upsilon)\varphi(T_{2b}) + [-\zeta_2(T_{2b}) + \zeta_2(\Upsilon)^2]\varphi(\Upsilon) + [-\zeta_2(T_{3e}) + 2\zeta_2(T_{2b})\zeta_2(\Upsilon) - \zeta_2(\Upsilon)^3]\varphi(|)
\end{aligned}$$

For the photon Green function:

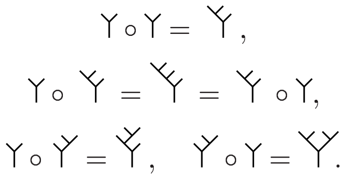
$$\bar\varphi_{\lambda\mu}(\Upsilon) = \varphi_{\lambda\mu}(\Upsilon) + \zeta_3(\Upsilon)\varphi_{\lambda\mu}(|)$$

$$\begin{aligned}
\bar\varphi_{\lambda\mu}(T_{2a}) &= \varphi_{\lambda\mu}(T_{2a}) + 2\zeta_3(\Upsilon)\varphi_{\lambda\mu}(\Upsilon) + \zeta_3(\Upsilon)^2\varphi_{\lambda\mu}(|)\\
\bar\varphi_{\lambda\mu}(T_{2b}) &= \varphi_{\lambda\mu}(T_{2b}) + \zeta_3(T_{2b})\varphi_{\lambda\mu}(|)
\end{aligned}$$

$$\begin{aligned}
\bar\varphi_{\lambda\mu}(T_{3a}) &= \varphi_{\lambda\mu}(T_{3a}) + 3\zeta_3(\Upsilon)\varphi_{\lambda\mu}(T_{2a}) + 3\zeta_3(\Upsilon)^2\varphi_{\lambda\mu}(\Upsilon) + \zeta_3(\Upsilon)^3\varphi_{\lambda\mu}(|)\\
\bar\varphi_{\lambda\mu}(T_{3b}) &= \varphi_{\lambda\mu}(T_{3b}) + \zeta_3(\Upsilon)\varphi_{\lambda\mu}(T_{2b}) + \zeta_3(T_{2b})\varphi_{\lambda\mu}(\Upsilon) + \zeta_3(T_{2b})\zeta_3(\Upsilon)\varphi_{\lambda\mu}(|)\\
\bar\varphi_{\lambda\mu}(T_{3c}) &= \varphi_{\lambda\mu}(T_{3c}) + \zeta_3(\Upsilon)\varphi_{\lambda\mu}(T_{2b}) - \zeta_3(T_{2b})\varphi_{\lambda\mu}(\Upsilon) + \zeta_3(T_{2b})\zeta_3(\Upsilon)\varphi_{\lambda\mu}(|)\\
\bar\varphi_{\lambda\mu}(T_{3d}) &= \varphi_{\lambda\mu}(T_{3d}) + \zeta_3(\Upsilon)\varphi_{\lambda\mu}(T_{2b}) + \zeta_3(T_{3d})\varphi_{\lambda\mu}(|)\\
\bar\varphi_{\lambda\mu}(T_{3e}) &= \varphi_{\lambda\mu}(T_{3e}) + \zeta_3(T_{3e})\varphi_{\lambda\mu}(|)
\end{aligned}$$

Algebra of photon and electron trees

We first describe the algebra structure on trees for the photon and for the electron propagators. Let $\circ : Y \times Y \longrightarrow Y$ be the associative product defined on each couple of trees (t, s) as the grafting $t \circ s$ of the tree s on the lefttmost leaf of t. Recursively: $| \circ s := s$ et $t \circ s := (t_l \circ s) \vee t_r$. (Remark that this product is the opposite of the operation "over" s/t described in [LR].) Examples of products are



The algebra of photons is $\mathcal{H}^\gamma := (\mathbb{C}Y, \circ)$. It is evidently generated by the trees of the form $| \vee t := V(t)$, with $t \in Y$, and therefore it is isomorphic to the tensor algebra $T(Y)$, with isomorphism $| \vee t \mapsto t$. In terms of Feynmann diagrams for the photon propagator, the product $\circ$ corresponds to the junction of two adiacent diagrams.

For the electron propagator, no inner product in the set of trees corresponds to the junction of Feynman graphs. We are then forced to consider the tensor product among trees: the algebra of electrons is, as before, the free associative algebra $\mathcal{H}^e := T(Y)$ on Y where the unit 1 is identified to the root $|$. Surprisingly, the generators with respect to the graph junction correspond to the elements $S^P(t) \in \mathcal{H}^e$, where S^P is the antipode of the pruning coproduct dual to the $\bullet$ product. In this context, S^P is an isomorphism ov vector spaces and can be simply regarded as a changement of basis in $\mathcal{H}^e$.

Finally, let $\mathcal{H}^{qed} := \mathcal{H}^\gamma \otimes \mathcal{H}^e$ be the algebra with componentwise product and unit $1 = | \otimes |$. We denote the elements $t \otimes s$ of $\mathcal{H}^{qed}$ by couples (t, s).

Hopf algebra for QED trees

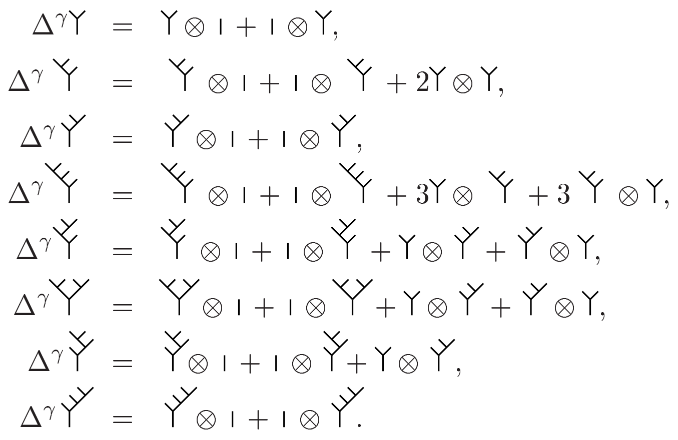
We now describe the Hopf algebra structure on $\mathcal{H}^{qed}$. For the photon trees, let $\Delta^\gamma : \mathcal{H}^\gamma \longrightarrow \mathcal{H}^\gamma \otimes \mathcal{H}^\gamma$ be the coproduct which cuts all the branches oriented like \, and puts them on the left of the tensor product. We can recursively define it as the algebra morphism such that $\Delta^\gamma(|) = | \otimes |$ and

$$\Delta^\gamma V(t) = V(t) \otimes | + F(V(t)),$$

where $F : \mathbb{C}Y \longrightarrow \mathbb{C}Y \otimes \mathbb{C}Y$ is an operator recursively defined by

$$\begin{aligned} F(|) &= | \otimes |, \\ F(V(t)) &= (Id \otimes V)F(t), \\ F(t \vee s) &= F(V(s))\Delta^\gamma t. \end{aligned}$$

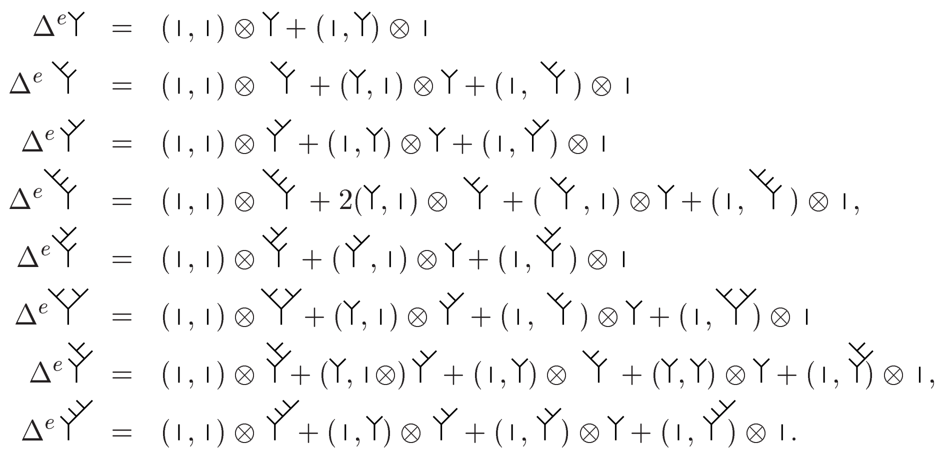
For instance :



For the electron trees, let $\Delta^e : \mathcal{H}^e \longrightarrow (\mathcal{H}^\gamma \otimes \mathcal{H}^e) \otimes \mathcal{H}^e$ be the coproduct which cuts this time all the branches, and remembers their orientation: \-branches produce a photon component, /-branches produce an electron component. Formally, we can define it as the algebra morphism such that $\Delta^e(\shortmid) = (\shortmid, \shortmid) \otimes \shortmid$ and

$$\Delta^e(t \vee s) = (\shortmid, t \vee s) \otimes \shortmid + \big((\circ \otimes Id) \otimes \vee\big)\Delta^\gamma(t)\Delta^e(s).$$

For instance :



Theorem 3.1 [BF2] *The coproduct $\Delta^{qed} : \mathcal{H}^{qed} \longrightarrow \mathcal{H}^{qed} \otimes \mathcal{H}^{qed}$ defined on generators $(t, s_1 ... s_n)$ as the product*

$$\Delta^{qed}(t, s_1 \ldots s_n) = \Delta^\gamma(t)\Delta^e(s_1) \ldots \Delta^e(s_n)$$

is coassociative and induces on $\mathcal{H}^{qed}$ the structure of a graded connected Hopf algebra, not commutative nor cocommutative, with counit $\epsilon(1) = 1$ and $\epsilon(t, s_1 \ldots s_n) = 0$.

Moreover, the photon algebra $\mathcal{H}^\gamma$ is a sub-Hopf algebra of $\mathcal{H}^{qed}$.

Renormalization via the Hopf algebra

Let $\langle f \otimes g, t \otimes s \rangle := f(t) \cdot g(s)$ be the canonical pairing between trees and maps on trees with values in an algebra.

Theorem 3.2 [BF1] *The coefficients $\bar{\varphi}_{\lambda\mu} : Y \longrightarrow M_4(\mathbb{C})$ for the expansion of the renormalized photon propagator, for massless renormalization, are given by*

$$\bar{\varphi}_{\lambda\mu}(t) \quad = \langle \zeta_3 \otimes \varphi_{\lambda\mu}, \Delta^\gamma(t) \rangle.$$

The coefficients $\bar{\varphi} : Y \longrightarrow M_4(\mathbb{C})$ for the expansion of the renormalized electron propagator, for massless renormalization, are given by

$$\bar{\varphi}(t) \quad = \langle \zeta \otimes \varphi, \Delta^e(t) \rangle,$$

where $\zeta : \mathcal{H}^{qed} \longrightarrow \mathbb{C}$ is the function defined as

$$\zeta(t, s_1 ... s_n) = \zeta_3(t)\zeta_2(S^P s_1)...\zeta_2(S^P s_n).$$

Conclusion

This theorem expresses the high similarity between the renormalization of scalar field theory, as described by the commutative Hopf algebra found by Connes and Kreimer on Feynman graphs, and the renormalization of vector-valued field theory, described by a non commutative Hopf algebra.

The main difference, beside being a non commutative setting, is peculiar to QED: the tree-expansion of the propagators reduces drastically the number of calculations and explicit counterterms.

In fact, it is known that some Feynman graphs of QED give rise to the same counterterms with opposite sign (Ward Identities). Hence, by summing up such graphs some counterterms cancel out mutually, and we finally find less summands then the original number. That's what happen to the renormalization performed on trees: the number of counterterms is considerably reduced.

However, this also turns out to be the step which brakes the proper renormalization: the Ward Identities cancel out too many contreterms of the graphs corresponding to each tree t, and the coefficients $\varphi(t)$, $\varphi_{\lambda\mu}(t)$ are not anymore necessarily finite!

To find finite renormalized coefficients, we are forced to sum up all the trees containg graphs related by the Ward Identities. One possible way is to sum up all trees with a given number (p, q) of oriented leaves $(\backslash, /)$: this corresponds to fix the number of photon and electron loops separately in the corresponding Feynman graphs.

Another way to find finite renormalized coefficients, the most economical, is to sum up all the trees with a given order n: this correspond to fix the number of total loops in the corresponding Feynman graphs. This leads to search the value of the coproducts Δ^γ and Δ^e on the elements $a_n = \sum_{|t|=n} t \in \mathbb{C}Y$, seen as elements of both algebras $\mathcal{H}^\gamma$ and $\mathcal{H}^e$.

The complete results can be found in [BF2]. Briefly, the renormalization of the photon propagator at the order of interaction gives rise to a noncommutative Hopf algebra $\mathcal{H}^{dif}$ which is a noncommutative version of the (commutative) part $\mathcal{H}^{CM}$ of the Connes-Moscovici Hopf algebra used in [CK]. Therefore, $\mathcal{H}^{dif}$ can be regarded as a "quantum group" for the group

$$G = \{ f : \mathbb{C} \longrightarrow \mathbb{C},\ f(x) = x + \sum_{n \geq 1} x^{n+1} a_n(f), a_n \in \mathbb{C} \}$$

of formal series such that $f(0) = 0$ and $f'(0) = 1$, considered with the composition law.

We sould remark, then, that formal series with noncommuting coefficients do not form a group with the composition, simply because the composition fails to be associative. However, the algebra of polynomial functions on such a group admits a noncommutative version: the coproduct on noncommuting variables is still coassociative!

The conclusion is then the following: the classical renormalization group fails to be a group for non-scalar field theories, but can be dually represented by a properly defined non-commutative Hopf algebra.

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Alessandra Frabetti
Institut de Mathématiques
Université de Lausanne
1015 Lausanne SWITZERLAND

e-mail: afrabett@magma.unil.ch
http://magma.unil.ch/afrabett/
tel: ++41 021 692 35 52
fax: ++41 021 692 35 55