Leibniz homology, relationship with Poisson manifolds, Gelfand-Fuchs cohomology and dialgebras

Note on the talk given at the II^{es} *Rencontres mathématiques de Glanon.*

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June 26, 1998

Abstract

This note is a brief survey on the main known results on Leibniz algebras, introduced by J.-L. Loday as a generalization of Lie algebras, and their relationship with two branches of classical geometry. First, we present two examples of a Leibniz algebra on multivector fields and differential forms on Poisson manifolds, found by Y. Kosmann-Schwarzbach. Secondly we introduce a Leibniz cohomology theory for differentiable manifolds and show its relationship, recently developped by J. Lodder, with the Gelfand-Fuchs cohomology of vector fields and some related invariants of foliations. Finally, we present the author's contribution to the computation of Leibniz homology, based on a generalization of associative algebras called dialgebras.

Introduction

Leibniz algebras are a generalization of Lie algebras introduced by J.-L. Loday in 1989. His main motivation was the search of an 'obstruction' to the periodicity of algebraic K-theory, which does not satisfy the Bott periodicity theorems valid for its topological version. We recall that cyclic homology of associative algebras can be regarded as an 'additive' or 'infinitesimal' version of rational algebraic K-theory, by virtue of the following results,

- $\bigwedge(K_*(A)_{\mathbf{Q}}) \cong H_*^{gr}(GL(A),\mathbf{Q}) \cong H_*^{Hoch}(\mathbf{Q}[GL(A)],\mathbf{Q})$, [Quillen]
- $\bigwedge(HC_{*-1}(A)) \cong H_*^{Lie}(\mathfrak{gl}(A),\mathbf{Q})$, [Loday-Quillen-Tsygan]

which hold for any unital associative algebra A. The long exact sequence

- $\ldots \longrightarrow HH_n(A) \longrightarrow HC_n(A) \longrightarrow HC_{n-2}(A) \longrightarrow \ldots$, [Connes]

leads us to interpret Hochschild homology as an obstruction to the periodicity of cyclic homology and to ask a natural question (see [L1] for more details): is there a theory KX_* (where X means 'unknown' for the moment) analogue to HH_*, which would be part of a spectral sequence involving the theory K_* and whose additive version would be Connes' long exact sequence?

The discovery of Leibniz algebras, and of their natural homology theory HL_*, produced a new relation

- $T(HH_{*-1}(A)) \cong HL_*(\mathfrak{gl}(A),\mathbf{Q})$, [Cuvier-Loday], where T denotes the tensor functor. Then the idea is to define $KX_*(A)$ in such a way that $T(KX_*(A)_{\mathbf{Q}}) \cong HX_*(\mathbf{Q}[GL(A)], \mathbf{Q})$, and then to compare it with $K_*(A)$. Here HX_* should be a new homology theory, related to Leibniz homology by a formula $HL_*(\mathfrak{g}, U\mathfrak{g}^{ad}) \cong HHX_*(UX\mathfrak{g})$, analogue to the classical relation

- $H_*^{Lie}(\mathfrak{g}, U\mathfrak{g}^{ad}) \cong HH_*(U\mathfrak{g})$,

which recovers the homology of Lie algebras from that of associative algebras. This program lead Loday to define a suitable 'associative' version of Leibniz algebras, called dialgebras, and to construct their natural homology theory. A final answer to the original problem is not achieved yet.

Beside this purely algebraic motivation and developement of Leibniz' setting, some relationships with classical geometry have recently been discovered, which could lead us to investigate Leibniz homology in view of concrete applications in non-commutative geometry and its physical interpretations.

This note is intended to be a brief summary of results on the current state of knowledge, in order to facilitate an approach to the subject as well as a deeper study of the role played by Leibniz algebras in classical geometry. It is organized as follows. In section 1 we give the basic definitions on Leibniz algebras and summarize the main properties of Leibniz homology and cohomology collected from the works of J.-L. Loday, T. Pirashvili and their team. In section 2 we illustrate two interesting examples of a Leibniz algebra constructed by Y. Kosmann-Schwarzbach on the spaces of multivector fields and differential forms on Poisson manifolds. In section 3 we recall the basics of Gelfand-Fuchs cohomology of vector fields on a differentiable manifold, and we present the Leibniz version of this theory, recently proposed by J. Lodder, and the relationship with the Godbillon-Vey invariant. Finally, in section 4, we present the 'associative' version of Leibniz algebras, called dialgebras, which can be regarded as one of the main tools for the computation of Leibniz homology, and the main results obtained by Loday and the author.

Contents

1 Leibniz algebras and (co)homology

The Chevalley-Eilenberg chain complex of a Lie algebra $\mathfrak{g}$ is the sequence of chain modules given by the exterior powers $\bigwedge^* \mathfrak{g}$ and boundary operators $d : \bigwedge^n \mathfrak{g} \longrightarrow \bigwedge^{n-1} \mathfrak{g}$ classically defined as

$$d(x_1 \wedge \ldots \wedge x_n) := \sum_{i<j} (-1)^{i+j+1} [x_i, x_j] \wedge x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n .$$

The property $d \circ d = 0$, which makes this sequence a chain complex, is proved by using the antisymmetry $x \wedge y = -y \wedge x$ of the exterior product, the Jacobi identity $[[x,y],z]+[[y,z],x]+[[z,x],y]=0$ and the antisymmetry $[x,y]=-[y,x]$ of the Lie bracket given on $\mathfrak{g}$.

In 1989, J.-L. Loday remarked that if we rewrite the Chevalley-Eilenberg boundary operator as

$$d(x_1 \wedge \ldots \wedge x_n) = \sum_{i<j} (-1)^{j+1} x_1 \wedge \ldots \wedge x_{i-1} \wedge [x_i, x_j] \wedge x_{i+1} \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_n,$$

then the property $d \circ d = 0$ is proved *without* making use of the antisymmetry properties of both the exterior product and of the Lie bracket on $\mathfrak{g}$: it suffices that the bracket satisfies the so called *Leibniz identity*

$$[x,[y,z]] = [[x,y],z] - [[x,z],y],$$

which is equivalent to the Jacobi identity any time the bracket is also antisymmetric.

From these considerations follow two facts:

- the Chevalley-Eilenberg boundary operator (in the Loday form) can be lifted to the *tensor powers* $\mathfrak{g}^{\otimes *}$ of a Lie algebra $\mathfrak{g}$, giving rise to a new chain complex $d : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}^{\otimes n-1}$ (called *Leibniz complex*);

- the Leibniz complex is well defined not only for Lie algebras, but more generally for the larger class of algebras whose bracket satisfies the Leibniz identity but is not necessarily antisymmetric. These algebras are called *Leibniz algebras*.

In conclusion, Loday established the following setup:

1.1 - Leibniz algebras. A *Leibniz algebra* is a vector space $\mathfrak{g}$ equipped with a binary operation $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies the *Leibniz identity*

$$(L) \quad [x,[y,z]] = [[x,y],z] - [[x,z],y], \qquad x,y,z \in \mathfrak{g}.$$

Lie algebras are a trivial example of Leibniz algebras, those with antisymmetric bracket. Non trivial examples are presented in section 2.

Some remarks on the name and the meaning of Leibniz identity may be helpful.

- It is custom to call "Leibniz rule" the property which characterizes the derivation of a product, namely $D([a,b]) = [D(a),b]+[a,D(b)]$. The Leibniz identity given above has in fact the same origin, since it says that *the adjoint operator* $\mathrm{ad}_x = [x,-]$ *is a derivation of the Lie bracket.*

- The Leibniz identity appears in the context of Lie algebras as the condition for which *any Lie algebra is a right module over itself*, when $x \in \mathfrak{g}$ is understood as an element of the module, and $y,z \in \mathfrak{g}$ as elements of the Lie algebra acting on the module from the right.

1.2 - Leibniz homology. Let $\mathfrak{g}$ be a Leibniz algebra. Consider the sequence of chain modules and boundary operators $d : \mathfrak{g}^{\otimes n} \longrightarrow \mathfrak{g}^{\otimes n-1}$, with

$$d(x_1 \otimes \ldots \otimes x_n) := \sum_{i<j} (-1)^{j+1} x_1 \otimes \ldots \otimes x_{i-1} \otimes [x_i,x_j] \otimes x_{i+1} \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_n.$$

The property $d \circ d = 0$ holds because of the Leibniz identity. Hence this sequence defines a chain complex whose homology is called *Leibniz homology* of $\mathfrak{g}$, and denoted by $HL_*(\mathfrak{g})$.

It is possible to define the notion of a *corepresentation* N of $\mathfrak{g}$ (bimodule satisfying suitable axioms, cf. [LP]) and to extend the Leibniz boundary operator to the sequence $d : N \otimes \mathfrak{g}^{\otimes n} \longrightarrow N \otimes \mathfrak{g}^{\otimes n-1}$, with

$$\begin{aligned} d(x_0 \otimes x_1 \otimes \ldots \otimes x_n) \;&:= \sum_{i<j} (-1)^{j+1} x_0 \otimes \ldots \otimes [x_i, x_j] \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_n \\ &+ [x_0, x_1] \otimes \ldots \otimes x_{n+1} + \sum_{i=1}^{n+1} (-1)^i [x_i, x_0] \otimes x_1 \otimes \ldots \otimes \hat{x}_i \otimes \ldots \otimes x_{n+1}, \end{aligned}$$

for $x_0 \in N$ and $x_1, \ldots, x_n \in \mathfrak{g}$. The homology of this new complex is called *Leibniz homology of* $\mathfrak{g}$ *with coefficients in* N, and denoted by $HL_*(\mathfrak{g}, N)$. The previous complex is recovered for any trivial corepresentation, for instance that on the ground field k, hence $HL_*(\mathfrak{g}) = HL_*(\mathfrak{g}, k)$.

The cohomology theory of Lie algebras can also be generalized to Leibniz algebras.

Recall that if $\mathfrak{g}$ is a Lie algebra, and M is a representation of $\mathfrak{g}$, the *Lie cohomology of* $\mathfrak{g}$ *with coefficients in* M is the cohomology $H^*_{Lie}(\mathfrak{g}, M)$ of the cochain complex $\delta : \mathrm{Hom}(\bigwedge^n \mathfrak{g}, M) \longrightarrow \mathrm{Hom}(\bigwedge^{n+1} \mathfrak{g}, M)$ with Chevalley-Eilenberg coboundary operator

$$\begin{aligned} (\delta f)(x_1 \wedge \ldots \wedge x_{n+1}) \;&:= \sum_{i<j} (-1)^{j+1} f(x_1 \wedge \ldots \wedge [x_i, x_j] \wedge \ldots \wedge \hat{x}_j \wedge \ldots \wedge x_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^i [f(x_1 \wedge \ldots \wedge \hat{x}_i \wedge \ldots \wedge x_{n+1}), x_i]. \end{aligned}$$

1.3 - Leibniz cohomology. Let $\mathfrak{g}$ be a Leibniz algebra and let M be a *representation* (bimodule) of $\mathfrak{g}$, i.e. a vector space endowed with a left and a right action of $\mathfrak{g}$, both denoted by $[\ ,\ ]$, satisfying the Leibniz identity (L) for any choice of each element x, y and z in M.

The cohomology theory dual to the homology $HL_*(\mathfrak{g})$ is defined by the sequence of cochain modules and coboundary operators $\delta : \mathrm{Hom}(\mathfrak{g}^{\otimes n}, M) \longrightarrow \mathrm{Hom}(\mathfrak{g}^{\otimes n+1}, M)$, with

$$\begin{aligned} (\delta f)(x_1 \otimes \ldots \otimes x_{n+1}) \;&:= \sum_{i<j} (-1)^{j+1} f(x_1 \otimes \ldots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \ldots \otimes \hat{x}_j \otimes \ldots \otimes x_{n+1}) \\ &+ [x_1, f(x_2 \otimes \ldots \otimes x_{n+1})] + \sum_{i=2}^{n+1} (-1)^i [f(x_1 \otimes \ldots \otimes \hat{x}_i \otimes \ldots \otimes x_{n+1}), x_i] \end{aligned}$$

(remark that $[x_1, m]$ is not necessarily equal to $-[m, x_1]$ in the Leibniz representation M). It is called *Leibniz cohomology of* $\mathfrak{g}$ *with coefficients in* M, and denoted by $HL^*(\mathfrak{g}, M)$.

1.4 - Main results on Leibniz (co)homology. Among the main properties of Leibniz homology and cohomology, here are the ones, proved by J.-L. Loday and T. Pirashvili in [L2] and [LP], which are expected to hold for a natural homology theory on Leibniz algebras.

- On the free Leibniz algebra $\mathrm{Leib}(V)$ on a vector space V, only the first homology group is non-zero, that is, $HL_1(\mathrm{Leib}(V)) \cong V$ and $HL_n(\mathrm{Leib}(V)) = 0$ for any $n > 1$.

- Leibniz homology indeed provides *a new family of invariants for Lie algebras*. It suffices to see it for an abelian Lie algebra $\mathfrak{g}$, which is a Lie algebra with zero bracket $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. All boundary operators are canonically zero, and yield the homology groups $H_n(\mathfrak{g}) = \bigwedge^n(\mathfrak{g})$ and $HL_n(\mathfrak{g}) = \mathfrak{g}^{\otimes n}$. In general, a representation M of a Lie algebra $\mathfrak{g}$ naturally determines a corepresentation M^{op} and a representation M of $\mathfrak{g}$ regarded as a

Leibniz algebra. Then Lie and Leibniz (co)homology of $\mathfrak{g}$ with values in M are related by natural morphisms

$$HL_*(\mathfrak{g}, M^{op}) \longrightarrow H_*^{Lie}(\mathfrak{g}, M),$$
$$H^*_{Lie}(\mathfrak{g}, M) \longrightarrow HL^*(\mathfrak{g}, M).$$

which are not necessarily injective nor surjective.

- *Leibniz homology and cohomology are derived functors* in the category of left modules over the universal enveloping algebra $UL(\mathfrak{g})$ of $\mathfrak{g}$. More precisely, if we denote by $\mathfrak{g}_{Lie}$ the Lie algebra obtained by identifying $[x,y]$ with $-[y,x]$, there are isomorphisms

$$HL_*(\mathfrak{g}, N) \cong \mathrm{Tor}_*^{UL(\mathfrak{g})}(N, U(\mathfrak{g}_{Lie})),$$
$$HL^*(\mathfrak{g}, M) \cong \mathrm{Ext}^*_{UL(\mathfrak{g})}(U(\mathfrak{g}_{Lie}), M),$$

where $UL(\mathfrak{g})$ is the unital ring which classifies $\mathfrak{g}$-representations as left $UL(\mathfrak{g})$-modules and $U(\mathfrak{g}_{Lie})$ is the universal enveloping algebra of the Lie algebra $\mathfrak{g}_{Lie}$.

- *The second cohomology group of $\mathfrak{g}$ with coefficients in M classifies the abelian extensions of the Leibniz algebra $\mathfrak{g}$ by M up to equivalence*, that is,

$$HL^2(\mathfrak{g}, M) \cong \mathrm{Ext}(\mathfrak{g}, M).$$

More specific results on Leibniz homology and cohomology can be found in [G], [L3], [LP], [NT], [O] and [P].

2 Example: Leibniz brackets extending a Poisson bracket

Consider a Poisson manifold (M, P), where M is a differentiable manifold and $P \in \Gamma^\infty(\bigwedge^2 TM)$ is the Poisson bivector, and let $\{\ ,\ \} : C^\infty(M) \otimes C^\infty(M) \longrightarrow C^\infty(M)$ be the Poisson bracket related to P by the formula $\{f, g\} = P(df, dg)$, where $d : C^\infty(M) \longrightarrow \Omega^1(M)$ is the first de Rham coboundary operator and the bivector P is seen as an antisymmetric operator acting on $\Omega^1(M) \otimes \Omega^1(M)$ with values in $C^\infty(M)$.

Let us describe two 'dual' settings of data on the Poisson manifold: the graded differential Lie algebra of multivector fields (with the Schouten-Nijenhuis bracket and the Lichnerowicz-Poisson differential), and the graded differential Lie algebra of differential forms (with the Koszul bracket and the de Rham differential). With these data one can define two canonical structures of a Leibniz algebra on multivector fields and on differential forms. Y. Kosmann-Schwarzbach, in [K-S], observed that both these Leibniz brackets extend the Poisson bracket on the functions, giving a new insight to the original Poisson structure.

2.1 - Graded differential Lie algebra of multivector fields. Let $V^n(M) := \Gamma^\infty(\bigwedge^n TM)$ be the space of *multivector fields* on M, for any $n \geq 1$, and set $V^0(M) := C^\infty(M)$. The Lie bracket $[\ ,\ ] : V^1(M) \otimes V^1(M) \longrightarrow V^1(M)$ canonically defined on the vector fields can be extended to a Lie bracket on multivector fields $[\ ,\ ] : V^n(M) \otimes V^m(M) \longrightarrow V^{n+m-1}(M)$, called *Schouten-Nijenhuis bracket*, cf. [N], [S], by setting

$$[X_1 \wedge \ldots \wedge X_n, Y_1 \wedge \ldots \wedge Y_m] := \sum_{i,j} (-1)^{i+j+1} [X_i, Y_j] \wedge X_1 \wedge \ldots \wedge \widehat{X}_i \wedge \ldots \wedge \widehat{Y}_j \wedge \ldots \wedge Y_m,$$

for $n, m \geq 1$. For $n, m = 0$ set $[f, g]_P = f \cdot g - g \cdot f = 0$, and for $n = 0$ and $m = 1$ (or the opposite) set $[f, X]_P := X(f)$. The Schouten-Nijenhuis bracket then satisfies the graded Jacobi identity, and it is antisymmetric.

Since M is a Poisson manifold, using the Schouten-Nijenhuis bracket and the Poisson bivector, we can furthermore introduce a differential operator $d_P : V^n(M) \longrightarrow V^{n+1}(M)$, called *Lichnerowicz-Poisson differential*, cf. [Li], by setting

$$d_P(V) := [P, V], \qquad V \in V^n(M).$$

This operator is a derivation of both the Schouten-Nijenhuis bracket and the exterior product, and $d_P \circ d_P = 0$. Hence $V^*(M)$ is a graded differential Lie algebra with $[\ ,\ ]$ and d_P.

2.2 - Graded differential Lie algebra of differential forms. Let $\Omega^n(M) := \Gamma^\infty(\bigwedge^n T^*M)$ be the spaces of *differential forms* on the manifold M, for $n \geq 1$, take $\Omega^0(M) := C^\infty(M)$, and let $d : \Omega^n(M) \longrightarrow \Omega^{n+1}(M)$ be the *de Rham differential*, which is a derivation of the exterior product of forms and an operator of zero square.

Using the Poisson bivector on the manifold M, we can introduce a Lie bracket $[\ ,\ ]_P : \Omega^n(M) \otimes \Omega^m(M) \longrightarrow \Omega^{n+m-1}(M)$ on differential forms, called *Koszul bracket*, cf. [K], in the following way. For $n, m = 0$ we set $[f, g]_P = f \cdot g - g \cdot f = 0$, for $n = 0$ and $m = 1$ (or the opposite) we set $[f, \omega]_P := P(df, \omega)$, for $n, m = 1$ we set $[\omega, \eta]_P := d(P(\omega, \eta))$ and finally, for $n > 1$ or $m > 1$, we extend this bracket with the suitable formula analogue to the one given in (2.1). The Koszul bracket satisfies the Jacobi identity and it is antisymmetric, and the de Rham differential turns out to be a derivation of it. Hence $\Omega^*(M)$ is a graded differential Lie algebra with d and $[\ ,\ ]_P$.

2.3 - Graded Leibniz algebra extending the Poisson bracket. Let M be a Poisson manifold as above. Then the Poisson bracket on differentiable functions can be recovered from

- the Schouten-Nijenhuis bracket and the Lichnerowicz-Poisson differential on multivector fields, because $\{f, g\} = -[f, d_P g]$ for any $f, g \in C^\infty(M)$;
- the Koszul bracket and the de Rham differential on differential forms, because $\{f, g\} = [f, dg]_P$ for any $f, g \in C^\infty(M)$.

These two identities have in common the formal shape (up to a sign), based on a *Lie bracket* and a *differential*, and the possibility of extending the involved operations from functions to higher orders multivector fields and differential forms. In fact, we can define two new binary operations

- $[U, V]_{d_P} := [V, d_P U]$ on multivector fields $U, V \in V^*(M)$,
- $[\omega, \eta]_{P,d} := [\omega, d\eta]_P$ on differential forms $\omega, \eta \in \Omega^*(M)$,

which coincide with the Poisson bracket on $C^\infty(M)$. These new brackets satisfy the graded Leibniz identity, suitable defined from (1.1), and in general they are *not* antisymmetric. Hence *they define on $V^*(M)$ and $\Omega^*(M)$ the structure of a graded Leibniz algebra which is not graded Lie.*

A detailed study of Leibniz invariants for these two classical algebras is in order.

2.4 - Derived Leibniz algebras. Remark that the procedure described above for multivector fields and differential forms holds more generally for any (graded) differential Lie algebra $(\mathfrak{g}, [\ ,\ ], d)$: the *derived bracket*

$$[x, y]_d := [x, dy], \qquad x, y \in \mathfrak{g},$$

is a (graded) Leibniz bracket which is not antisymmetric, in general. The new algebra $(\mathfrak{g}, [\ ,\ ]_d)$ is called *derived Leibniz algebra* and denoted by $\mathfrak{g}_d$.

3 Leibniz version of Gelfand-Fuchs cohomology

3.1 - Gelfand-Fuchs cohomology. Let M be a differentiable manifold and let $V(M) := \Gamma^\infty(TM)$ be the real Lie algebra of vector fields on M (V^1 in the notations of section 2). Fix on $V(M)$ the usual C^∞-topology corresponding to uniform convergence of any derivative on compact sets of M, and consider the *continuous* Lie cochain complex of $V(M)$ with coefficients in the real trivial representation, with the Chevalley-Eilenberg coboundary operator

$$\delta : \mathrm{Hom}^{cont}(\bigwedge^n V(M), \mathbf{R}) \longrightarrow \mathrm{Hom}^{cont}(\bigwedge^{n+1} V(M), \mathbf{R}),$$

described in section 1. The cohomology of this complex is called *Gelfand-Fuchs cohomology of* $V(M)$, and denoted by $H^*_{GF}(V(M), \mathbf{R})$ or simply $H^*_{GF}(V(M))$. Many computations of Gelfand-Fuchs cohomology groups have been done, mainly in the 70's, by Gelfand and Fuchs, Losik, Guillemin, Trauber, Haefliger, Bott and Segal. We recall the main results.

- If M is a compact manifold and we only consider the space $V_{comp}(M) \subset V(M)$ of vector fields with compact support, then $H^*_{GF}(V_{comp}(M))$ is a finite dimensional graded algebra, [GF].

- More generally, if M is a compact manifold or the interior of a compact manifold with boundary, then $H^*_{GF}(V(M)) \cong H^*(\mathrm{E}(M))$, where $\mathrm{E}(M) = \Gamma^\infty(E_M)$ are the sections of a certain fibre bundle E_M on M associated to its tangent bundle, such that each fibre is an open manifold with the same cohomology as $V(\mathbf{R}^n)$, cf. [BS]. Hence the computation of H^*_{GF} for $V(M)$ can be done by patching together local results on open subsets of M, i.e. for $V(\mathbf{R}^n)$.

- The computation can be furthermore restricted to the space W_n of formal vector fields on the origin of $\mathbf{R}^n$, because of the isomorphism $H^*_{GF}(V(\mathbf{R}^n)) \cong H^*_{Lie}(W_n)$, cf. [BS]. Moreover, the product of any two elements of $H^*_{GF}(V(\mathbf{R}^n))$ is always zero.

- The algebra $H^*_{Lie}(W_n)$ is finite dimensional and $H^q_{Lie}(W_n) = 0$ for $0 < q \leq 2n$ and $q > n(n+2)$. More precisely, from the computation of the cohomology of W_n with coefficients in the adjoint representation $W'_n := \mathrm{Hom}(W_n, \mathbf{R})$, we get an isomorphism, cf. [GF],
$$H^q_{Lie}(W_n, W'_n) \cong H^{2n+1}_{Lie}(W_n, \mathbf{R}) \otimes H^{q-2n}_{Lie}(\mathfrak{gl}_n(\mathbf{R})).$$
Hence the first non-zero cohomology group of W_n is $H^{2n+1}_{Lie}(W_n) \cong H^{2n}_{Lie}(W_n, W'_n)$.

The Gelfand-Fuchs cohomology of vector fields on a manifold M is related to the invariants of certain foliations on M. Here is the main application, due to Haefliger [H].

- Let M and X be smooth manifolds. A smooth foliation F on $M \times X$ transverse to the fibres at the projection on M determines uniquely a continuous morphism of graded algebras

$$\mathrm{char}(\mathrm{F}) : H^*_{GF}(V(X)) \longrightarrow H^*_{dR}(M),$$

which is called *characteristic morphism* of the foliation F. (Here $H^*_{dR}(M)$ denotes the de Rham cohomology of M.) It is functorial in F and depends only on the homotopy class of F.

- In particular, when $X = \mathbf{R}^n$ and F is a smooth foliation on M of codimension n, we get a uniquely determined characteristic morphism

$$\mathrm{char}(\mathrm{F}) : H^*_{GF}(V(\mathbf{R}^n)) \cong H^*_{Lie}(W_n) \longrightarrow H^*_{dR}(M).$$

3.2 - Continuous Leibniz cohomology. Let $\mathfrak{g}$ be a topological Leibniz algebra and V a topological $\mathfrak{g}$-representation. In the complex of Leibniz cochains given in (1.3) we can consider the submodules of continuous cochains $\mathrm{Hom}^{cont}(\mathfrak{g}^{\otimes n}, V) \subset \mathrm{Hom}(\mathfrak{g}^{\otimes n}, V)$. Since the Leibniz differential df of any continuous cochain f is still continuous, the sequence of continuous cochains forms a subcomplex whose cohomology is called *continuous Leibniz cohomology of* $\mathfrak{g}$ *with coefficients in* V and still denoted by $HL^*(\mathfrak{g}, V)$.

In particular, let M be a differentiable manifold and let its Lie algebra of vector fields $V(M) = \Gamma^\infty(TM)$ be equipped with the C^∞-topology. Then the *continuous Leibniz cohomology of* $V(M)$ *with trivial coefficients*, denoted by $HL^*(V(M), \mathbf{R})$ or $HL^*(V(M))$, is the cohomology of the cochain complex with cochain modules

$$C^n(V(M), \mathbf{R}) := \mathrm{Hom}^{cont}(V(M)^{\otimes n}, \mathbf{R}),$$

and Leibniz coboundary operator $\delta : C^n(V(M), \mathbf{R}) \longrightarrow C^{n+1}(V(M), \mathbf{R})$ given by

$$(\delta f)(X_1 \otimes \ldots \otimes X_{n+1}) = \sum_{i<j} (-1)^{j+1} f(X_1 \otimes \ldots \otimes X_{i-1} \otimes [X_i, X_j] \otimes X_{i+1} \otimes \ldots \otimes \hat{X}_j \otimes \ldots \otimes X_{n+1}).$$

J. Lodder showed in [Lo] some similarities and some differences between Gelfand-Fuchs cohomology and continuous Leibniz cohomology:

- The Leibniz cohomology groups $HL^*(V(M))$ are diffeomorphism invariant.

- For $M = \mathbf{R}^n$, it suffices to compute the Leibniz cohomology of the formal vector fields on the origin of $\mathbf{R}^n$, because $HL^*(V(\mathbf{R}^n)) \cong HL^*(W_n)$. Unlike for Gelfand-Fuchs cohomology, there exist infinite families of elements whose products are non-zero.

- The graded algebra $HL^*(W_n)$ is finite dimensional, and can be described in terms of the classical variational map $\mathrm{var} : \Omega^q(W_n) \longrightarrow \Omega^{q-1}(W_n; W_n')$,

$$\mathrm{var}(\alpha)(X_1 \wedge \ldots \wedge X_{q-1})(X_0) := (-1)^{q-1} \alpha(X_0 \wedge X_1 \wedge \ldots \wedge X_{q-1}).$$

More precisely, there is an isomorphism of dual Leibniz algebras

$$HL^*(W_n) \cong \mathbf{R} \oplus [\mathrm{im(var)} \oplus \mathrm{coker(var)}[1]) \otimes T(\mathrm{coker(var)}[1] \oplus \ker(\mathrm{var})[-1]),$$

where T denotes the tensor algebra, the symbol [1] after a graded space shifts its grading by 1 and similarly by -1 for $[-1]$.

Moreover $HL^{2n+1}(W_n) \cong H^{2n+1}_{Lie}(W_n)$, hence the first obstruction to the homotopy invariance of $HL^*(W_n)$ is the universal Godbillon-Vey invariant in dimension $2n+1$.

3.3 - Characteristic Leibniz homomorphism For what concerns applications of continuous Leibniz cohomology of vector fields, Lodder conjectures in [Lo] the existence of a *characteristic Leibniz homomorphism*

$$L_{\mathrm{F}} : HL^*(V(\mathbf{R}^n)) \cong HL^*(W_n) \longrightarrow HL^*(V(M), C^\infty(M)),$$

associated to any smooth foliation F of codimension n on M, with the property of giving rise to a commutative diagram

$$\begin{array}{ccc} H^*_{Lie}(W_n) & \xrightarrow{\mathrm{char}(\mathrm{F})} & H^*_{dR}(M) \\ \downarrow & & \downarrow \\ HL^*(W_n) & \xrightarrow{L_{\mathrm{F}}} & HL^*(V(M), C^\infty(M)) \end{array} .$$

In this diagram, $HL^*(V(M), C^\infty(M))$ denotes the *continuous Leibniz cohomology of $V(M)$ with coefficients in the space of differentiable functions $C^\infty(M)$*, which is a representation of the Leibniz algebra of vector fields, in the sense of (1.3), with the natural actions

$$[X, f] := X(f) \qquad \text{and} \qquad [f, X] := -[X, f].$$

The map $H^*_{Lie}(W_n) \longrightarrow HL^*(W_n)$ is canonically defined for any Lie algebra, cf. (1.3), while the map $H^*_{dR}(M) \longrightarrow HL^*(V(M), C^\infty(M))$ is described in next paragraph.

3.4 - De Rham cohomology as Lie $C^\infty(M)$-cohomology of vector fields. It is well known that differential forms $\Omega^n(M) := \Gamma^\infty(\bigwedge^n T^*M)$ on a manifold M can be coupled with multivector fields $V^n(M) := \Gamma^\infty(\bigwedge^n TM) \cong \bigwedge^n V^1(M)$ (here $V^1(M) = V(M)$), and yield differentiable functions on the manifold. Moreover, this coupling is continuous and linear in both variables with respect to the left action of $C^\infty(M)$ on $V^n(M)$ and $\Omega^n(M)$, for any $n \geq 0$. In other words, differential forms can be seen as continuous $C^\infty(M)$-linear maps acting on multivector fields,

$$\Omega^n(M) \cong \mathrm{Hom}^{cont}_{C^\infty(M)}(\bigwedge^n V^1(M), C^\infty(M)).$$

The de Rham coboundary operator $d : \Omega^n(M) \longrightarrow \Omega^{n+1}(M)$ can be read through this isomorphism, and it turns out that it acts as a Chevalley-Eilenberg coboundary operator, cf. [Sp]

$$\begin{aligned} d\omega(X_1 \wedge \ldots \wedge X_{n+1}) &= \sum_{i<j} (-1)^{j+1} \omega(X_1 \wedge \ldots \wedge [X_i, X_j] \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_{n+1}) \\ &+ \sum_{i=1}^{n+1} (-1)^{i+1} X_i(\omega(X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_{n+1})). \end{aligned}$$

Hence, *the de Rham cohomology of M is the restriction of the continuous Lie cohomology of vector fields on M, with coefficients in $C^\infty(M)$, to the category of left modules over $C^\infty(M)$.*

The map $H^*_{dR}(M) \longrightarrow HL^*(V(M), C^\infty(M))$ is then induced by the composition of the maps

$$\begin{array}{ccc} \Omega^n(M) \cong \mathrm{Hom}^{cont}_{C^\infty(M)}(\bigwedge^n V(M), C^\infty(M)) & \longrightarrow & \mathrm{Hom}^{cont}(\bigwedge^n V(M), C^\infty(M)) \\ & & \downarrow \\ & & \mathrm{Hom}^{cont}(V(M)^{\otimes n}, C^\infty(M)) \end{array}$$

at the cochain level.

3.5 - Remark. The Leibniz coboundary of an n-tensor is not necessarily an $(n+1)$-tensor. The above observation leads us to consider the possibility of introducing the non-commutative version of de Rham cohomology given by the continuous Leibniz cohomology of vector fields with coefficients in differentiable functions, in the category of left $C^\infty(M)$-modules.

A key observation of J. Lodder, however, makes it impossible: while the differential $d\omega$ of an alternating form $\omega \in \mathrm{Hom}^{cont}_{C^\infty(M)}(\bigwedge^{\otimes n} V(M), C^\infty(M))$ is $C^\infty(M)$-linear, *the differential $d\omega$ of a tensor form $\omega \in \mathrm{Hom}^{cont}_{C^\infty(M)}(V(M)^{\otimes n}, C^\infty(M))$ is not $C^\infty(M)$-linear.* Hence the continuous Leibniz cochain complex of $V(M)$ with coefficients in $C^\infty(M)$ can not be restricted to the category of left $C^\infty(M)$-modules.

4 Computations and dialgebra homology

The computation of Leibniz homology in general bears the same difficulties as that of Lie homology. In the classical setting, some results are known when the Lie algebra is obtained from an associative algebra by means of the canonical bracket $[a,b] := a\cdot b - b\cdot a$. In particular, for the Lie algebra $\mathfrak{gl}_n(A)$ of matrices with coefficients in an associative algebra A, or better for the inductive limit $\mathfrak{gl}(A) = \mathfrak{gl}_\infty(A)$, the following fundamental theorem, due to Loday, Quillen and Tsygan, [LQ], [T], enables us to determine the Lie homology in terms of the cyclic homology of the algebra A (which is easier to compute): for any unital associative algebra A over a field of characteristic zero there is an isomorphism of graded Hopf algebras

$$H_*(\mathfrak{gl}(A)) \cong \bigwedge(HC_{*-1}(A)).$$

We then expect that Leibniz homology could be also computed in terms of some new homology theory. This leads us to search for an 'associative version' of Leibniz algebras, together with its 'natural' and 'cyclic' homology theory.

4.1 - Dialgebras. A *dialgebra* is a vector space D equipped with two operations, $\dashv, \vdash : D \otimes D \longrightarrow D$, called left and right products, satisfying the following five axioms:

$$(D) \quad \begin{cases} a \dashv (b \dashv c) = (a \dashv b) \dashv c = a \dashv (b \vdash c), \\ (a \vdash b) \dashv c = a \vdash (b \dashv c), \\ (a \dashv b) \vdash c = a \vdash (b \vdash c) = (a \vdash b) \vdash c, \end{cases} \qquad a,b,c \in D.$$

Any associative algebra A is a trivial example of a dialgebra, for which the left and right products coincide, that is, $a \dashv b := a\cdot b =: a \vdash b$ for $a,b \in A$.

The motivation for considering the axioms (D) is that any dialgebra D determines uniquely a Leibniz algebra D_L, called *canonical Leibniz algebra of D*, whose bracket

$$[a,b] := a \dashv b - b \vdash a, \qquad a,b \in D,$$

is not antisymmetric, unless the left and right products coincide. (Observe that the choice of the axioms (D) is not unique.) The assignement $D \mapsto D_L$ yields a commutative diagram of categories and functors

$$\begin{array}{ccc} \mathbf{Alg}_k & \xrightarrow{\ -\ } & \mathbf{Lie}_k \\ \downarrow & & \downarrow \\ \mathbf{Dig}_k & \xrightarrow{\ -\ } & \mathbf{Leib}_k \end{array} .$$

4.2 - Examples.

1) Derived dialgebras. A non-trivial example of a dialgebra is constructed on an associative differential algebra (A, d), by introducing the *derived products*

$$a \dashv b := a \cdot d(b), \qquad a \vdash b := d(a) \cdot b, \qquad a, b \in A$$

(with appropriate signs in the case of graded algebras), which satisfy the axioms (D) because $d(a\ db) = da\ db = d(da\ b)$. The triple $A_d := (A, \dashv, \vdash)$ is called *derived dialgebra* of (A, d). The canonical Leibniz algebra of the derived dialgebra A_d then coincides with the derived Leibniz algebra of the differential Lie algebra (A_L, d), as defined in (2.4), i.e. $(A_d)_L \cong (A_L)_d$.

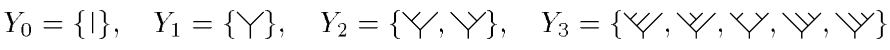
2) Dialgebra induced by a bimodule morphism. In [L4], Loday exhibits a typical example of a dialgebra, constructed on a bimodule M over an associative algebra A. If there is a morphism of A-bimodules $f : M \longrightarrow A$, then M is a dialgebra with products

$$m \dashv m' := m \cdot f(m'), \qquad m \vdash m' := f(m) \cdot m', \qquad m, m' \in M.$$

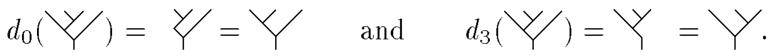
In [F4] it is shown how this procedure can be inverted without restricting hypothesis, so all dialgebras can be constructed in this way.

The natural dialgebra boundary operator $d : D^{\otimes n} \longrightarrow D^{\otimes n-1}$ was introduced by Loday, on the hint that the composition $D^{\otimes 3} \longrightarrow D^{\otimes 2} \longrightarrow D^{\otimes 1}$ yields zero because of the relations (D). The chain module in dimension 3 must be 5 copies of $D^{\otimes 3}$, one for each equality of (D), and that in dimension 2 must be 2 copies of $D^{\otimes 2}$, one for each operation. In dimension n, it turns out to be $c_n = \frac{2n!}{n!(n+1)!}$ copies of $D^{\otimes n}$, where c_n is known as *Catalan number* and has many relevant combinatorical interpretations.

4.3 - The chain complex of planar binary trees. In order to define face maps on the chain modules of a dialgebra D, the best way to label the c_n copies of $D^{\otimes n}$ is to consider c_n as the cardinality of the set Y_n of *planar binary trees* with one root and $n+1$ leaves. Examples of such sets are:



The family of planar binary trees $\{Y_n\}$ is a pre-simplicial set with face maps $d_i : Y_n \longrightarrow Y_{n-1}$ which delete the i^{th} leaf of the trees, for any $i = 0, \ldots, n$. For example,



For any field k, the boundary operator $d := \sum_{i=1}^{n-1} (-1)^i d_i$ satisfies $d \circ d = 0$, hence it defines a chain complex $d : k[Y_n] \longrightarrow k[Y_{n-1}]$.

4.4 - Dialgebra homology and cohomology. Let D be a dialgebra over a field k. The boundary operator defined on trees can be extended to a chain complex

$$d : k[Y_n] \otimes D^{\otimes n} \longrightarrow k[Y_{n-1}] \otimes D^{\otimes n-1}$$

by setting the face maps, for $i = 1, \ldots, n-1$, as

$$d_i(y \otimes (a_1 \otimes \ldots \otimes a_n)) := d_i(y) \otimes (a_1 \otimes \ldots \otimes a_i \circ_i^y a_{i+1} \otimes \ldots \otimes a_n),$$

where $\circ_i^y := \dashv$, if the i^{th} leaf of y is oriented as $\backslash$ and $\circ_i^y := \vdash$, if the i^{th} leaf of y is oriented as $/$. This operator still satisfy $d \circ d = 0$, because of the axioms (D). The homology of this complex is called *dialgebra homology of D with trivial coefficients*, and denoted by $HY_*(D, k)$ or $HY_*(D)$.

The dialgebra chain complex can also be defined with coefficients in a *corepresentation* N of the dialgebra D, that is, a bimodule over D with actions characterized by suitable relations. It suffices to define the extra face maps d_0, d_n that, for simplicity of the reader, we omit here (details can be found in [F4]). The homology of the complex

$$d : k[Y_n] \otimes N \otimes D^{\otimes n} \longrightarrow k[Y_{n-1}] \otimes N \otimes D^{\otimes n-1}$$

is called *dialgebra homology of D with coefficients in N*, and denoted by $HY_*(D, N)$.

Similarly, we can introduce a cochain complex with coboundary operator

$$\delta : \mathrm{Hom}(k[Y_n] \otimes D^n, M) \longrightarrow \mathrm{Hom}(k[Y_{n+1}] \otimes D^{n+1}, M),$$

where M is a *representation* of D, that is, a bimodule over D with four actions (two left actions $\dashv$, $\vdash$ and two right actions $\dashv$, $\vdash$) satisfying the relations (D) for any possible choice of a, b or c in M. The cohomology of this complex is called *dialgebra cohomology of D with coefficients in M*, and denoted by $HY^*(D, M)$. (Details on dialgebra cohomology can be found in [F4].)

4.5 - Main results on dialgebra (co)homology. The main properties of dialgebra homology and cohomology are proved in [L5] and [F4]:

- On the free dialgebra $\mathcal{D}ias(V)$ on a vector space V only the first homology group is non-zero, that is, $HY_1(\mathcal{D}ias(V)) \cong V$ and $HY_n(\mathcal{D}ias(V)) = 0$ for any $n > 1$.

- Dialgebra homology *is a generalization of Hochschild homology*, because

  $$HY_n(A, M) \cong H_n(A, M),$$

  for any H-unital associative algebra A and any A-bimodule M.

- *The chain complex of bar-unital dialgebras with trivial coefficients is acyclic*, that is,

  $$HY_n(D) = 0, \qquad n \geq 1.$$

  Here 'bar-unital' means that the dialgebra is equipped with a *bar-unit* (the best notion of unit which can be given to a dialgebra): an element 1 which is a unit only on the bar-side of the products, $a \dashv 1 = a = 1 \vdash a$ for any $a \in D$.

- *Dialgebra homology and cohomology are derived functors* in the category of left modules over the universal enveloping algebra $E(D)$ of a dialgebra D. More precisely,

  $$HY_n(D, N) \cong \mathrm{Tor}_n^{E(D)}(N, M(D)),$$
  $$HY^n(D, N) \cong \mathrm{Ext}^n_{E(D)}(M(D), M),$$

  where $M(D)$ is a suitable D-representation which coincide with the algebra itself when $D = A$ is an associative algebra.

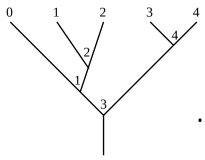
- *The second cohomology group classifies the abelian extensions of dialgebras up to equivalence*, that is,

  $$HY^2(D, M) \cong \mathrm{Ext}(D, M),$$

  for any dialgebra D and any D-representation M.

So, dialgebra homology is the natural generalization of Hochschild homology to the dialgebra setting. However, following the classical philosophy, in order to compute the Leibniz homology of the matrices $\mathfrak{gl}(D)$, with coefficients in a dialgebra D, it is necessary to find also a generalization of cyclic homology of associative algebras.

4.6 - The chain complex of permutations. Let S_n be the set of permutations on n elements. This set is in bijection with the set $\tilde{Y}_n$ of *leveled planar binary trees* with one root and $n+1$ leaves. A tree is called *with level* if each of its n internal vertices is placed in one of the n levels of height (from the root to the leaves). For example, the tree is not leveled, while and are trees with levels. To describe the bijection $S_n \simeq \tilde{Y}_n$, it suffices to choose a rule to order the internal vertices of leveled trees, for instance, by giving to each vertex the same label as the leaf at the right hand side of the hole which is closed by the vertex, as in the example below:



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An n-tree with level is then identified with the permutation $\sigma \in S_n$ which associates to each labeled vertex its level. For instance, the tree of the example above is associated to the permutation $\binom{1\ 2\ 3\ 4}{2\ 3\ 1\ 4} = (123)$.

The face maps defined on planar binary trees can be easily extended to the trees with levels, and hence to the permutations: for all $i = 0, \ldots, n$ there exist face maps $d_i : S_n \longrightarrow S_{n-1}$ such that the operator $d = \sum_{i=0}^{n} (-1)^i d_i$ satisfies $d \circ d = 0$. Hence there is a chain complex $d : k[S_n] \longrightarrow k[S_{n-1}$.

4.7 - Symmetric homology. Let D be a dialgebra over a field k, and let N be a corepresentation of D. The boundary operator defined on permutations can be suitably extended to a chain complex

$$d : k[S_n] \otimes N \otimes D^{\otimes n} \longrightarrow k[S_{n-1}] \otimes N \otimes D^{\otimes n-1},$$

using the orientation of the leaves of the leveled tree associated to the permutation, which has the same form as the one given in (4.4). The new homology is called *symmetric homology of D with coefficients in N* and denoted by $HS_*(D,N)$. In particular, the symmetric homology of D with coefficients in itself is denoted by $HHS_*(D)$.

4.8 - Main results on symmetric homology. The symmetric homology is not proved to be a derived functor, nor to be the natural homology theory for a new kind of algebras. Its main application is the following theorem, proved in [F2] and [O]:

- *The Leibniz homology of dialgebras of infinite matrices is computed in terms of the symmetric homology.* More precisely, for any bar-unital dialgebra D there is an isomorphism of graded dual Leibniz algebras

$$HL_*(\mathfrak{gl}(D)) \cong T(HHS_{*-1}(D)).$$

- *The symmetric homology is also a generalization of Hochschild homology*, in fact for any unital associative algebra A and any unital bimodule M aver A we have

$$HS_n(A,M) \cong H_n(A,M).$$

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