Algèbre/*Algebra*

Dialgebra homology of associative algebras

Alessandra FRABETTI

Abstract – Let A be an associative algebra, and M a bimodule over A. Consider A as a dialgebra (see [5]), with left and right products equal to the given associative multiplication. We define the dialgebra homology $HY_*(A,M)$ of A with coefficients in M, and show, when A is unital, that it is isomorphic to the Hochschild homology $H_*(A,M)$. It is a consequence of the combinatorial properties of the set of binary trees.

Homologie de digèbres pour les algèbres associatives

Résumé – Soit A une algèbre associative et M un bimodule sur A. On considère A comme une digèbre (cf. [5]), avec produits gauche et droit qui coincident avec le produit associatif donné. On définit l'homologie $HY_*(A,M)$ de A à coefficients dans M, et on montre, quand A est unitaire, qu'elle est isomorphe à l'homologie de Hochschild $H_*(A,M)$. C'est une conséquence des propriétés combinatoires de l'ensemble des arbres binaires.

Version française abrégée – Soit k un corps commutatif de caractéristique zéro. J.-L. Loday a introduit, dans [5], une généralization des algèbres associatives appellée digèbres. Une *digèbre* D sur le corps k est un k-espace vectoriel avec deux produits associatifs $\dashv$, $\vdash$ qui satisfont à trois axiomes supplémentaires [5]. Une algèbre associative est une digèbre pour laquelle les deux produits coincident. Pour les digèbres, Loday a aussi introduit une théorie d'homologie naturelle $HY(D)$ à coefficients dans le D-module trivial k, qui a été étendue dans [7] à une théorie d'homologie $HY(D,M)$ à coefficients dans un D-module non trivial M. Dans cette note on considère l'homologie des digèbres $HY(A,M)$ pour une algèbre associative A.

Soit A une algèbre associative sur k et M un bimodule sur A. L'homologie de Hochschild de A à coefficients dans M est l'homologie $H_*(A,M)$ du complexe associé au module pré-simplicial $C(A,M)=\{C_n(A,M)=M\otimes A^{\otimes n}\}$, muni des faces $d_i : M\otimes A^{\otimes n}\longrightarrow M\otimes A^{\otimes n-1}$, qui satisfont aux relations simpliciales

$$(i)\quad d_i d_j = d_{j-1} d_i \quad \text{pour } i<j$$

et donnent le bord de Hochschild $b=\sum_{i=0}^{n}(-1)^i d_i$.

1 Ensemble pseudo-simplicial des arbres binaires. – Soit Y_n l'ensemble des arbres planaires binaires avec $n+1$ feuilles et une racine. Pour tout n-arbre y, soit $d_i(y)$ le $(n-1)$-arbre que on obtient en effaçant la $i^{ème}$ feuille de y $(0\leq i\leq n)$, et soit $s_j(y)$ le $(n+1)$-arbre avec la $j^{ème}$ feuille $|$ remplacé par l'arbre $\curlyvee$. Les applications $d_i : Y_n \longrightarrow Y_{n-1}$ et $s_j : Y_n\longrightarrow Y_{n+1}$ satisfont aux relations (i) et

$$(ii)\quad d_i s_j = \begin{cases} s_{j-1} d_i & \text{pour } i<j \\ id & \text{pour } i=j, i=j+1 \\ s_j d_{i-1} & \text{pour } i>j+1 \end{cases} .$$

Donc la famille $Y = \{Y_n\}_{n\geq 0}$ forme un ensemble *pseudo-simplicial* [2], et on peut considérer le complexe $k[Y_*]$ des arbres binaires, avec opérateur bord $d : k[Y_n] \longrightarrow k[Y_{n-1}]$, $d = \sum_{i=0}^{n}(-1)^i d_i$.

2 Définition. – L'*homologie* HY *de* A *à coefficients dans* M est l'homologie du complexe $CY_*(A, M)$ associée au produit des modules pré-simpliciaux $k[Y]$ et $C(A, M)$. On la note par $HY_*(A, M)$. Le module des n-chaines est

$$CY_n(A, M) := k[Y_n] \otimes M \otimes A^{\otimes n},$$

et les faces sont

$$d_i : CY_n(A, M) \longrightarrow CY_{n-1}(A, M), \quad d_i(y \otimes \underline{a}) = d_i(y) \otimes d_i(\underline{a}).$$

Quand $M = k$ est le module trivial, l'homologie $HY_*(A, k)$ est précisément l'homologie $HY_*(A)$ introduite dans [5].

Etant donné une algèbre associative A et un A-bimodule M, on peut facilement démontrer que le complexe de Hochschild $C_*(A, M)$ est un sous-complexe de $CY_*(A, M)$, et que $HY_0(A, M) = H_0(A, M)$ et $HY_1(A, M) = H_1(A, M)$.

3 Théorème. – *Supposons que* A *ait une unité, et que* M *soit un bimodule unitaire. Alors pour tout* $n \geq 0$ *on a un isomorphisme*

$$HY_n(A, M) \cong H_n(A, M)$$

entre l'homologie de digèbre et l'homologie de Hochschild de A.

Ce théorème est une conséquence des deux résultats suivants.

4 Lemme. – *Soient* M *et* N *deux modules pseudo-simpliciaux. Soient* M_* *et* N_* *les deux complexes de chaines associés, et* $(M \times N)_*$ *le complexe associé au produit des deux modules pseudo-simpliciaux. Si* M_* *est acyclique, alors on a un isomorphisme*

$$H_n((M \times N)_*) \cong H_n(N_*),$$

pour tout $n \geq 0$.

5 Proposition. – *Le complexe de chaines des arbres binaires* $k[Y_*]$ *est acyclique.*

Démonstration du théorème 3. Quand A est muni d'une unité, le complexe de Hochschild $C_*(A, M)$ est muni des dégénérescences

$$s_j(a_0, a_1, ..., a_n) := (a_0, ..., a_j, 1, a_{j+1}, ..., a_n), \quad \text{pour } j = 0, ..., n,$$

qui satisfont aux relations (ii). (En plus $C(A, M)$ devient un module simplicial.) Puisque $k[Y_*]$ est acyclique, le théorème suit du lemme 4.

La famille des arbres binaires $Y = \{Y_n\}_{n\geq 0}$ peut être remplacée par la famille des groupes symétriques $S = \{S_n\}_{n\geq 0}$, qui forment un module pseudo-simplicial et donne une autre théorie d'homologie $HS_*(A, M)$. J. Lodder a montré dans [6] que le complexe associé $k[S_*]$ est acyclique, donc de la même façon que pour 3 on a le théorème suivant.

6 Théorème. – *Supposons que* A *ait une unité, et que* M *soit un bimodule unitaire. Alors pour tout* $n \geq 0$ *on a un isomorphisme*

$$HS_n(A, M) \cong H_n(A, M).$$

English Version – Let k be a field with characteristic zero. J.-L. Loday introduced in [5] a generalization of associative algebras called dialgebras. A *dialgebra* D over the field k is a k-vector space with two associative products $\dashv, \vdash$ satisfying three more axioms. An associative algebra is a dialgebra for which the two products coincide. For dialgebras, Loday also introduced a natural homology theory $HY(D)$ with coefficients in the trivial D-module k, which was extended in [7] to a homology theory $HY(D,M)$ with coefficients in a non-trivial D-module M. In this note we restrict our attention to the dialgebra homology $HY(A,M)$ for an associative algebra A with coefficients in an A-bimodule M.

First we fix a terminology on some useful variations of simplicial sets.

7 Pseudo-simplicial sets. – Recall that a *simplicial* set is a collection $E = \{E_n, n \geq 0\}$ of sets endowed with face maps $d_i : E_n \longrightarrow E_{n-1}$, for $i = 0, ...n$, and degeneracies $s_j : E_n \longrightarrow E_{n+1}$, for $j = 0, ...n$, satisfying the following relations

$$\begin{array}{llcl}
(i) & d_i d_j & = & d_{j-1} d_i \quad \text{for } i < j \\
(ii) & d_i s_j & = & \begin{cases} s_{j-1} d_i & \text{for } i < j \\ id & \text{for } i = j, i = j+1 \\ s_j d_{i-1} & \text{for } i > j+1 \end{cases} \\
(iii) & s_i s_j & = & s_{j+1} s_i \quad \text{for } i \leq j.
\end{array}$$

By *pre-simplicial* set we mean a collection $E = \{E_n, n \geq 0\}$ of sets endowed *only* with face maps d_i satisfying the above relations (i). By *pseudo-simplicial* set we mean a pre-simplicial set E endowed with *pseudo*-degeneracies s_j satisfying relations (ii) but *not* relations (iii) (see [2], [3]).

Recall also that degeneracies are not necessary to define a chain-complex structure on a simplicial module. In other words, any pre-simplicial set $E = \{E_n\}_{n \geq 0}$ gives rise to a chain complex of k-modules. Take as chain module of dimension n the free k-module generated by the set E_n, and define the boundary operator $d : k[E_n] \longrightarrow k[E_{n-1}]$ as the alternated sum of the face maps d_i, i.e. $d = \sum_{i=0}^{n} (-1)^i d_i$.

8 Hochschild pre-simplicial module. Let A be an associative algebra over k and M a bimodule over A. The Hochschild complex of A with coefficients in M, denoted $C_*(A,M)$, is the complex associated to the pre-simplicial module $C(A,M) = \{C_n(A,M) = M \otimes A^{\otimes n}\}$ with face maps $d_i : M \otimes A^{\otimes n} \longrightarrow M \otimes A^{\otimes n-1}$,

$$d_i(\underline{a}) = \begin{cases} (a_0, \ldots, a_i \cdot a_{i+1}, \ldots, a_n) & \text{, for } i = 0, ..., n-1, \\ (a_n \cdot a_0, \ldots, a_{n-1}) & \text{, for } i = n, \end{cases}$$

for $\underline{a} = (a_0, a_1, \ldots, a_n)$ with $a_0 \in M$ and $a_1, ..., a_n \in A$. Thus the Hochschild boundary is $b = \sum_{i=0}^{n} (-1)^i d_i$. Note that, without further assumptions on A and M, the pre-simplicial module $C(A,M)$ is *not* pseudo-simplicial. In other words, it is not endowed with degeneracy maps satisfying relations (ii).

9 Pseudo-simplicial set of binary trees. Let Y_n be the set of planar binary trees with $n+1$ leaves and a root. For any n-tree y, let $d_i(y)$ be the $(n-1)$-tree obtained by deleting the i-th leaf from y. Then the maps $d_i : Y_n \longrightarrow Y_{n-1}$ satisfy the simplicial relations (i). (In fact on both sides the maps act on a tree by deleting the i-th and the j-th leaf, in the two possible orders. It is equivalent to delete them simultaneously.) Similarly, for any n-tree y, let $s_j(y)$ be the $(n+1)$-tree obtained from y by *bifurcating* the j-th leaf, i.e. by substituting the j-th leaf $|$ with the tree $\curlyvee$. Then the maps $s_j : Y_n \longrightarrow Y_{n+1}$ satisfy relations (ii). Thus the family $Y = \{Y_n\}_{n\geq 0}$ form a pseudo-simplicial set and there is a well defined chain-complex $k[Y_*]$ of binary trees

$$k[Y_0] \longleftarrow k[Y_1] \longleftarrow k[Y_2] \longleftarrow ... \longleftarrow k[Y_n] \longleftarrow ....$$

Remark that the family $\{Y_n\}_{n\geq 0}$ of binary trees is in fact a particular pseudo-simplicial set (almost simplicial!), since the degeneracy maps $s_j : Y_n \longrightarrow Y_{n+1}$ satisfy also the relations (iii) for which $i < j$.

10 Definition. – The *dialgebra homology of A with coefficients in M* is the homology of the complex $CY_*(A, M)$ associated to the product of the pre-simplicial modules $k[Y]$ and $C(A, M)$. It is denoted by $HY_*(A, M)$. The module of n chains is

$$CY_n(A, M) := k[Y_n] \otimes M \otimes A^{\otimes n},$$

and the face maps are

$$d_i : CY_n(A, M) \longrightarrow CY_{n-1}(A, M), \quad d_i(y \otimes \underline{a}) = d_i(y) \otimes d_i(\underline{a}).$$

(The comparison between this complex and the tensor product $k[Y_*] \otimes C_*(A, M)$ is given by the Alexander-Whitney map [1].) In particular, when $M = k$ is the trivial module, the homology $HY_*(A, k)$ is precisely the homology $HY_*(A)$ introduced in [5], and when $A = M = k$ is the ground field, we recover the complex $k[Y_*]$ of binary trees.

The homology $HY_*(A, M)$ is a particular case of the homology $HY_*(D, N)$ defined in [7] for a dialgebra D with coefficients in a D-corepresentation N. In the general case, the complex $CY_*(D, N)$ is *not* the product of two pre-simplicial modules, since the analogue Hochschild faces d_i on a dialgebra already depend on trees.

Given an associative algebra A and an A-bimodule M, it is easy to check that the Hochschild complex $C_*(A, M)$ is a sub-complex of $CY_*(A, M)$, with inclusions

$$i : C_n(A, M) \hookrightarrow CY_n(A, M), \quad \underline{a} \mapsto \curlyvee \otimes \underline{a}.$$

It follows easily that $HY_0(A, M) = H_0(A, M)$ and $HY_1(A, M) = H_1(A, M)$.

11 Theorem. – *Let A be an associative algebra and M an A-bimodule. Suppose that A has a unit, and M is a unitary bimodule (i.e. $1 \cdot m = m \cdot 1 = m$ for any $m \in M$). Then for any $n \geq 0$ there is an isomorphism*

$$HY_n(A, M) \cong H_n(A, M)$$

between the dialgebra homology and the Hochschild homology of A.

This theorem is a consequence of the following lemma about pseudo-simplicial modules and the combinatorial properties of the set of binary trees, cf. proposition 14.

12 Lemma. – *Let M and N be two pseudo-simplicial modules. Denote by M_* and N_* the two associated chain-complexes, and by $(M \times N)_*$ the chain-complex associated to the product of the two pseudo-simplicial modules. If M_* is acyclic, then we have an isomorphism*

$$H_n((M \times N)_*) \cong H_n(N_*),$$

for any $n \geq 0$.

This lemma follows directly from the following result (Theorem 1.4 in [3]):

13 Proposition. – (Inasaridze [3]) *Let A_{**} be a double pseudo-simplicial abelian group. Then there exists a spectral sequence*

$$E^2_{pq} = H^v_p H^h_q(A_{**}) \Longrightarrow H_{p+q}(\Delta A_*),$$

where $\Delta A_n = A_{nn}$ denotes the diagonal complex.

14 Proposition. – *The chain complex $k[Y_*]$ of binary trees is acyclic.*

Proof. Define a map $r : Y_n \longrightarrow Y_{n+1}$ by

$$r(y) :=$$ 

One can show that r is an extra-degeneracy, i.e. it satisfies the relations $d_0 r = id$ and $d_i r = r d_{i-1}$ for any $i > 0$. Then, as for simplicial modules, the identity $dr + rd = id$ follows, i.e. the map $r : k[Y_n] \longrightarrow k[Y_{n-1}]$ is a homotopy between id and 0.

Proof of theorem 11. When A is endowed with a unit, the Hochschild complex $C_*(A, M)$ is endowed with the well known degeneracies

$$s_j(a_0, a_1, ..., a_n) := (a_0, ..., a_j, 1, a_{j+1}, ..., a_n), \quad \text{for } j = 0, ..., n,$$

which satisfy relations (ii) and (iii) for all $i \leq j$. Thus $C(A, M)$ becomes a simplicial module and, a fortiori, a pseudo-simplicial module. Since $k[Y_*]$ is acyclic, the theorem follows from lemma 12.

15 Variation HS_* of dialgebra homology. The pseudo-simplicial set of binary trees $\{Y_n\}_{n\geq 0}$ can be sligthly modified by considering rooted planar binary trees *with levels*. A tree with levels has all its internal vertices placed at different levels of height from the root. Since any n-tree has precisely n internal vertices (and thus n possible levels where they can be placed) there exists a bijective correspondence between the internal vertices, previously labeled, and their levels. This bijection gives a permutation on n elements.

Let S_n denote the set of permutations on n elements. We clearly have a surjective map $S_n \longrightarrow Y_n$ which identifies a permutationwith a leveled tree and then forgets the levels. The face maps $d_i : Y_n \longrightarrow Y_{n-1}$ and the degeneracy maps $s_j : Y_n \longrightarrow Y_{n+1}$ defined in 9 clearly still make sense on trees with levels. Let us keep, for the moment, the same notation. Thus we obtain a pseudo-simplicial set of permutations $S = \{S_n\}_{n\geq 0}$, whose associated chain-complex

$$k[S_0] \longleftarrow k[S_1] \longleftarrow k[S_2] \longleftarrow ... \longleftarrow k[S_n] \longleftarrow ...,$$

is denoted by $k[S_*]$.

Let A be an associative algebra, M an A-bimodule and $C(A, M)$ the Hochschild pre-simplicial module (see 8). We define the homology $HS_*(A, M)$ as the homology of the complex $CS_*(A, M)$ associated to the product of the pre-simplicial modules $k[S]$ and $C(A, M)$.

The homology $HS_*(A, M)$ is a particular case of a homology theory $HS_*(D, N)$ which can be defined for any dialgebra D and any D-corepresentation N. This homology theory crops up when one computes the Leibniz homology of the Leibniz algebra of matrices (cf. [8]).

J. Lodder proved in [6] that the complex $k[S_*]$ is acyclic. Thus, after lemma 12, we obtain the following

16 Theorem. – *Let A be an associative algebra and M an A-bimodule. Suppose that A has a unit, and M is a unitary bimodule. Then for any $n \geq 0$ there is an isomorphism*

$$HS_n(A, M) \cong H_n(A, M)$$

between the homology HS and the Hochschild homology of A.

Remark that, when the bimodule M of coefficients is A itself, a different proof is given by Cuvier in [4].

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Dipartimento di Matematica
dell'Università di Bologna,
Piazza di Porta San Donato, 5
40127 Bologna, Italy
e-mail : frabetti@dm.unibo.it